Chapter 1

Introduction

Since it was founded by Aristotle, Logic has been the science devoted to the laws of the correct reasoning. Traditionally, one of the fundamental laws has been the Bivalence Principle, which states that every proposition is either true or false, independently how difficult might be in some cases to determine its truth value. This traditional logic under the Bivalence Principle, that we call *Classical Logic*, turned out to be an excellent tool for the mathematical work, specially after Mathematical Logic was born in the nineteenth century with Augustus de Morgan, George Boole and Gottlob Frege among others. It is not strange, since Mathematics do use precise concepts and always works with statements that are intended to be either true or false.

Nevertheless, Aristotle already noticed that many of the concepts that are commonly used outside the strict mathematical discourse are far from being precise; on the contrary they refer to qualities that admit degrees. In *Categories* 8, 10b 26-32 he writes:

> Qualifications admit of a more and a less; for one thing is called more pale or less pale than another, and more just than another. Moreover, it itself sustains increase (for what is pale can still become paler) – not in all cases though, but in most. It might be questioned whether one justice is called more a justice than another, and similarly for the other conditions.[1]

Some lines below (*Categories* 8, 11a 2-5) he adds:

> At any rate things spoken of in virtue of these unquestionably admit of a more and a less: one man is called more grammatical than another, juster, healthier, and so on. Triangle and square do not seem to admit of a more, nor does any other shape.

This kind of predicates appear in propositions that often do not seem neither completely true nor completely false. This is the vagueness phenomenon and it becomes a real challenge for Logic when one considers some reasonings that involve vague predicates, such as the so called *Sorites Paradox*:

Premises:

(1) A man who has twenty thousand hairs on his head is not bald.

[1] We cite from the English translation in [6].

(2) If a man who is not bald loses one hair, he is still not bald.

Conclusion:

(3) A man with no hair on his head is not bald.

This reasoning is a paradox because it seems to be correct (one can derive the conclusion from the premisses by using Modus Ponens twenty thounsand times), and it derives a clearly false conclusion from (apparently) true premisses (we have no doubt that (1) is true, and (2) seems also true); something that can never happen in a correct reasoning.

Several solutions to cope with the vagueness phenomena have been proposed (see e.g. [139]). One of them is Fuzzy Logic. As other non-classical logics, it rejects the Bivalence Principle and proposes infinitely-valued logics to substitute Classical Logic. Historically this idea comes from the theory of Fuzzy Sets proposed in 1965 by Lotfi Zadeh in [140]. His idea consisted on modeling vague predicates with fuzzy sets, i.e. sets where the objects can belong in a greater or lesser degree. Formally, a fuzzy set is a pair $\langle X, \mu \rangle$ where X is a set (in the classical sense) and $\mu : X \to [0, 1]$ is a function (called *membership function*) that maps every object $x \in X$ to a real number $\mu(x)$ between 0 and 1, that is interpreted as the degree of membership of the object in the fuzzy set. For instance, given the vague predicate *tall*, one can define a fuzzy set by considering: $X := [0.3, 2.4]$ (set of possible heights in meters) and the membership function:

$$\mu(x) = \begin{cases} 0 & \text{if } x < 1.2, \\ \frac{5}{3}x - 2 & \text{if } 1.2 \leq x \leq 1.8, \\ 1 & \text{if } x > 1.8. \end{cases}$$

This fuzzy set models the predicate *tall* in such a way that people whose height is more than 1.8 meters are definitely tall, people with less than 1.2 meters of height are definitely not tall and people with an intermediate height is given the quality *tall* in an intermediate degree according to a linear function.[2] This interpretation amounts to a certain use of infinitely-many truth-values in the following way: if some individual a in our universe of discourse has a height $x \in X$, then we say that the sentence 'a is tall' is true at degree $\mu(x)$.

However, if one wants to model vague predicates in terms of fuzzy sets, some way to combine them is required, since one will want to consider propositions where vague predicates are combined in disjunctions, conjunctions, negations and other usual logical connectives. For instance, to model in a truth-functional way the conjunction of two vague predicates one can consider their corresponding fuzzy sets and provide some binary function such that, given any object and its membership degrees to the fuzzy sets, returns the membership degree to the intersection. In the same way, a binary function is needed for the union of fuzzy sets, and a unary function for the complement. In his foundational papers

[2] Of course, the choice of the membership function must depend on the context in which one wants to model the vague predicate. For instance, the meaning of *tall* is not the same when the universe of individuals are basketball players or just ordinary people.

Zadeh proposed the functions $\min\{x, y\}$, $\max\{x, y\}$ and $1 - x$ respectively for the intersection, the union and the complement of fuzzy sets.

In 1969 Goguen proposes in [75] a solution to the Sorites Paradox in terms of fuzzy sets, but using some other functions to combine them, namely the truth-functions introduced by Łukasiewicz for some infinitely-valued logics. Therefore, a few words on the origin of many-valued logics are needed here.

The first many-valued logic was introduced in 1918 by Jan Łukasiewicz. It was a three-valued logic proposed to deal with the problem of future contingents. According to Łukasiewicz, the Bivalence Principle implies a kind of determinism, since it forces all propositions to be either true or false, including those that state some facts about the future.[3] He believes that it is more intuitive to claim that these propositions are still neither true nor false, and thus he introduces a new truth-value for them that he calls *possible*. If $\{0, \frac{1}{2}, 1\}$ is the set formed by his three truth-values, Łukasiewicz defines the logical connectives of implication and negation by means of the following tables:

$\rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

	$\neg$
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

Therefore, he is using the Principle of Extensionality to compute the truth-value of every complex proposition from the truth-values of its parts by means of the tables. The meaning of the tables can be almost explained in the following way. Suppose that $\{T, F\}$ are the classical truth-values, *true* and *false*. Then, the three Łukasiewicz's truth-values can be interpreted as sets of these classical truth-values: $0 = \{F\}$, $1 = \{T\}$ and $\frac{1}{2} = \{T, F\}$, since the value $\frac{1}{2}$ is given to those proposition about future facts which still we do not know whether they will be true or false. Almost all the values in the tables are obtained by operating all the elements in these sets according to the classical connectives of implication and negation. For instance, $\{T, F\} \rightarrow \{T\} = \{T\}$, because according to the classical implication we have $T \rightarrow T = T$ and $F \rightarrow T = T$. There is only one exception which is not compatible with this interpretation: $\frac{1}{2} \rightarrow \frac{1}{2}$ should be $\frac{1}{2}$, but it is 1. The obvious reason seems to be that Łukasiewicz wanted to preserve the validity of the Identity Law, $\varphi \rightarrow \varphi$.

In 1922 Łukasiewicz generalizes the three-valued logic to an n-valued logic for every $n \geq 4$ finite, where the set of truth-values is $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$, and the logical operations are defined by $x \rightarrow y := \min\{1, 1-x+y\}$ and $\neg x := 1-x$.

[3]The problem of future contingents can be also traced back to Aristotle; viz. his famous naval battle example.

Finally, in 1930, with Alfred Tarski in [110], they generalize it to an infinitely-valued logic where the set of truth-values is the real unit interval and the truth-functions are defined as in the finitely-valued case. Several additional connectives are defined in the following way: $x\&y := \neg(x \to \neg y)$, $x \vee y := (x \to y) \to y$ and $x \wedge y := \neg(\neg x \vee \neg y)$. Then, some properties of the classical conjunction split between & and $\wedge$:

1. For every a, b, c, $a\&b \leq c$ iff $a \leq b \to c$ (residuation law)
2. For every a, b, $a \to b = 1$ iff $a \wedge b = a$ iff $a \leq b$ ($\wedge = \min$)

By using these truth-functions, Goguen was able to propose a solution to the sorites paradox. Indeed, he considered that *bald* is a vague predicate and thus it must be interpreted by a fuzzy set. (1) can be given the minimum truth-value, since a man with twenty thousand hairs on his head is not bald, i.e. he belongs to the set of bald men at degree 0. However, (2) is not absolutely true. Consider that its truth-value is $\frac{19999}{20000}$, almost 1. Let v_i be the truth-value of 'A man with i hairs in his head is not bald'. Then, we have $v_{20000} = 1$ and $v_i \to v_{i-1} = \frac{19999}{20000}$. Interpreting $\to$ with Łukasiewicz implication and after an easy computation, it comes out that $v_0 = 0$, hence the paradox disappears.

Afterwards, following the truth-functional setting of Zadeh, some other functions were proposed to model the combination of fuzzy sets. They were required to satisfy certain conditions. For instance, the functions for disjunction and conjunction were required to be associative and commutative. Alsina, Trillas and Valverde (see e.g. [4]) proposed a class of functions taken from the theory of probabilistic metric spaces (see [133, 134]), the triangular norms (t-norms, for short), to model the intersection of fuzzy sets, their dual functions, the t-conorms for the unions, and the so-called weak negation functions for the complement (see [136, 48]). T-norms are binary operations defined on the real unit interval which are associative, commutative, monotonic and have 1 as neutral element. As regards to implication, mainly two kinds were proposed:

S-implications: those satisfying that for every a, b, $a \to b = \neg a \vee b$, and
R-implications: those satisfying the residuation law.

In [137] R-implications were chosen in order to deal with the Modus Ponens rule. For every continuous t-norm there is an associated R-implication, which is called its *residuum*. Although the continuity was not required in the definition of t-norm, the majority of the known examples were actually continuous, for instance the minimum (the original interpretation for the intersection of fuzzy sets in Zadeh's seminal paper), the Łukasiewicz interpretation of the connective & (we will call it *Łukasiewicz t-norm*) and the product of reals (we will call it *product t-norm*). Moreover, all continuous t-norms can be represented as an ordinal sum of these three examples, as proved in [118] and in [108].

Interestingly, two of the three basic continuous t-norms and their residua had been used in the semantics of some infinitely-valued logics before fuzzy sets were defined. On the one hand, the Łukasiewicz t-norm, as we have already explained, appeared in the semantics of Łukasiewicz's infinitely-valued logic. He also defined a Hilbert-style calculus whose axioms were:

(Ł1) $\varphi \to (\psi \to \varphi)$

(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(Ł3) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

(Ł4) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

(Ł5) $((\varphi \to \psi) \to (\psi \to \varphi)) \to (\psi \to \varphi)$

and Modus Ponens was the only inference rule. Let $\mathcal{A}$ be the algebra defined over $[0, 1]$ by the Łukasiewicz truth-functions. He conjectured that the tautologies of the infinitely-valued logic given by the matrix $\langle \mathcal{A}, \{1\} \rangle$ coincide with the theorems of this Hilbert-style calculus. However, he was not able to prove it. In 1935 Mordchaj Wajsberg claimed that he had found a proof, but he never gave it. The conjecture was finally proved by syntactical means in 1958 by Rose and Rosser [131] and algebraically in 1959 by Chang [25, 26]. It is important to remark that Meredith showed in [113] the redundancy of the axiom (Ł5) and that Hay improved the result in 1963 (in [86]) by proving that the Hilbert-style calculus in fact coincides with the finitary fragment of Łukasiewicz logic.

On the other hand, the minimum t-norm also corresponds to the semantical interpretation of the conjunction in a many-valued logic. Indeed, Gödel used it in [74] to study some linearly ordered matrix semantics for superintuitionistic logics. Dummett gave in [47] a sound and complete Hilbert-style calculus for the infinitely-valued logic corresponding to the matrix defined over $[0, 1]$ by the minimum t-norm and its residuated implication.

These two examples suggested that, with the usage of t-norms and their residua, Fuzzy Logic was very close to the apparently independent field of many-valued logics. The next step in approaching both fields was done in [83] when Hájek, Godo and Esteva gave also a Hilbert-style calculus for the remaining prominent example of continuous t-norm: the product t-norm.

Therefore, it became clear that at least some part of Fuzzy Logic was directly related to the study of some many-valued logics. This led to the distinction by the founder of the field, Zadeh, between a wide and a narrow sense of 'Fuzzy Logic'. In [141] he writes:

> The term 'Fuzzy Logic' has two different meanings: wide and narrow. In a narrow sense, fuzzy logic, FLn, is a logical system which aims at a formalization of approximate reasoning. In this sense, FLn is an extension of multivalued logic. However, the agenda of FLn is quite different from that of traditional multivalued logics. In particular, such key concepts in FLn as the concept of a linguistic variable, canonical form, fuzzy if-then rule, fuzzy quantification and defuzzification, predicate modification, truth qualification, the extension principle, the compositional rule of inference and interpolative reasoning, among others, are not addressed in traditional systems. This is the reason why FLn has a much wider range of applications than traditional systems. In its wide sense, fuzzy logic, FLw, is fuzzily synonymous with the fuzzy set theory, FST, which is the theory of classes with unsharp boundaries. FST is much broader than FLn and includes the latter as one of its branches.

Once it was shown that the logics defined by the three main continuous t-norms and their residua enjoyed a syntactical calculus, Hájek proposed in [79] a new logical system, that he called Basic Fuzzy Logic (BL, for short) to capture the logic given by the class of all continuous t-norms.[4] His conjecture was proved in [80, 30]. Nevertheless, Esteva and Godo noticed that the necessary and sufficient condition for a t-norm to have a residuum was not the continuity, but the left-continuity. Thus, they wanted to find the fuzzy logic corresponding to the bigger class of all left-continuous t-norms. To fulfil this aim, they proposed a new Hilbert-style calculus in [51] called Monoidal T-norm based Logic (MTL, for short). MTL was proved to be indeed the logic of all left-continuous t-norms and their residua by Jenei and Montagna in [100]. Thus, in a sense MTL can be considered the real basic fuzzy logic, since it is the weakest logic which is complete with respect to a semantics given by a class of t-norms and their residua (this kind of logics are called *t-norm based fuzzy logics* or just *t-norm based logics*). The word 'Monoidal' in the name is due to the fact that MTL is an axiomatic extension of another many-valued logic proposed by Höhle in [87] called *Monoidal Logic* (ML, for short). ML has been proved to be equivalent to a contraction-less substructural logic, namely the system $\mathrm{H}_{\mathrm{BCK}}$ (also called FL_{ew}) of Ono and Komori (see [127, 126]). Therefore, since MTL enjoys neither the contraction rule, it can be regarded not only as a fuzzy logic and as a many-valued logic, but also as a substructural logic.

In this dissertation, that pertains to Fuzzy Logic in narrow sense, we study MTL (and its axiomatic extensions) from the two first points of view of the previous list: as a fuzzy logic and as a many-valued logic. More than that, we study MTL as an *algebraizable* many-valued logic. Indeed, traditionally many algebraic counterparts have been used in the research on many-valued logics. For instance, Chang introduced MV-algebras to study Łukasiewicz logic[5], a subclass of Heyting algebras called *G-algebras* have been introduced for Gödel-Dummett logic, residuated lattices (in the sense of [43]) for ML and so on. Hájek gave an algebraic semantics for BL, namely the variety of BL-algebras, a subclass of residuated lattices. A bigger variety of residuated lattices has been used to algebraize MTL by Esteva and Godo, the variety of MTL-algebras $\mathbb{MTL}$. Actually, MTL is algebraizable in the sense of Blok and Pigozzi (see [19]) and $\mathbb{MTL}$ is its equivalent quasivariety semantics. Therefore, all its finitary extensions are also algebraizable and their algebraic semantics are the corresponding subquasivarieties of $\mathbb{MTL}$. When we restrict to axiomatic extensions, the corresponding algebraic semantics are the subvarieties of $\mathbb{MTL}$. Moreover, the algebraizability implies a number of results connecting algebraic properties of the semantics with logical properties, the so-called *bridge theorems* of Abstract Algebraic Logic (see [62]). A relevant subclass of MTL-algebras are those defined over the real unit interval, which are exactly those where the conjunction & is interpreted by a left-continuous t-norm. These algebras are called *standard*. For every finitary extension of MTL a crucial question, from the Fuzzy Logic point of view, is whether it is complete w.r.t. the standard algebras of its corresponding quasi-

[4]Łukasiewicz, Gödel-Dummett and Product logics were proved to be axiomatic extensions of BL.

[5]A polinomially equivalent algebraic semantics for Łukasiewicz logic, the class of Wajsberg algebras, has been proposed in [130, 63].

variety. This kind of result is called *standard completeness*.[6]

Many investigations in the algebraic direction for MTL and its finitary extensions have been already carried out (see e.g. [30, 100, 49, 53, 3, 32, 33, 81, 71, 72, 116, 88, 89, 90, 69, 91]). Thanks to these works (and others) some parts of the lattice of subvarieties of $\mathbb{MTL}$ are already known, namely all varieties of MV-algebras (see [103]), all varieties of G-algebras (see e.g. [76]), many varieties of BL-algebras (see [3, 53]) and some other parts of the lattice (see e.g. [71, 72]). The results obtained in BL strongly relied on the knowledge on its linearly ordered algebras, since all BL-algebras (in fact, all MTL-algebras) are representable as subdirect product of linearly ordered ones, and linearly ordered BL-algebras were well described in terms of ordinal sums of some basic components (generalizing the corresponding result for continuous t-norms). Unfortunately, such a representation is not known for linearly ordered MTL-algebras (also called *MTL-chains*). Thus, the structure of the lattice of varieties of MTL-algebras was very far from being known.

With this dissertation we want to contribute to the task of describing axiomatic extensions of t-norm based fuzzy logics, or equivalently, varieties of MTL-algebras and their properties. On the one hand, we try to describe the structure of MTL-chains. Our first attempt consists in using the decomposition as ordinal sum of indecomposable semihoops. However, although we prove that for every chain exists a maximum decomposition in this sense, we show that the class of indecomposable semihoops is really huge and it seems too difficult to describe. Nevertheless, we study a significant class of indecomposable semihoops: those satisfying a weak form of cancellation. The second attempt uses one of the methods proposed by Jenei to built left-continuous t-norms, the so-called connected rotation-annihilation construction. We show that for every chain there is also a maximal decomposition as connected rotation-annihilation. The task of describing the indecomposable chains seems also too far away, but we can still study some cases of this decomposition, which leads to the theory of perfect and bipartite MTL-algebras. On the other hand, we focus on several varieties of MTL-algebras satisfying some nice properties, namely the varieties of n-contractive algebras. They seem a good choice because they correspond to the axiomatic extensions of MTL satisfying the global Deduction-Detachment Theorem, and they contain all the locally finite varieties of MTL-algebras, in particular the weak nilpotent minimum algebras (a subvariety that we study in a greater detail). For all the varieties of MTL-algebras defined in the thesis we study several relevant logical and algebraic properties, mainly standard completeness properties, local finiteness, finite embedding property, finite model property and decidability.

[6]The logics studied in this thesis belong to another interesting class of logics studied by general methods, namely the class of Weakly Implicative fuzzy logics (see [38]). This class is claimed to be the right class of fuzzy logics (in narrow sense) in [12]. Using the general results mentioned above the authors obtain completeness w.r.t. linearly ordered algebras, local Deduction-Detachment Theorem, subdirect decomposition theorem, etc. Moreover, we also would like to point out that axiomatic extensions of MTL are *core fuzzy logics* in the sense of [82].

Finally, we consider another aspect of Fuzzy Logic in narrow sense: which should be the use of the intermediate truth-values? In MTL and its finitary extensions, these truth-values do not seem to be used in a deep way, since in their algebraization the only distinguished value is the top element, as in classical logic. For this reason, following Pavelka's idea in [128] for Łukasiewicz logic, we consider expansions of t-norm based logics with constants for the intermediate truth-values, allowing them to play an explicit role in the language. The originality of our proposal lies in carrying out an algebraic approach to these expansions, i.e. studying them also as algebraizable logics.

The thesis is structured in eleven chapters. After this introduction, a first group of chapters (from the second till the fifth) are introductory, with necessary preliminaries, sometimes interpreted from our point of view; the rest contains our contributions to the field.

1. In the second chapter we introduce the basic notions and notation that will be used throughout all the dissertation on Universal Algebra and Algebraic Logic.

2. In the third chapter the logic MTL, the main subject of the study, is formally introduced. The three perspectives (as a substructural logic, as an algebraizable many-valued logic and as a t-norm based fuzzy logic) for MTL are also formally discussed.

3. In the fourth chapter we present the basic results and definitions for MTL-algebras that we will need to study their varieties. In particular, the decomposition of every algebra as a sudirect product of chains is proved and some useful methods to build new IMTL-algebras are introduced, namely Jenei's rotation and rotation-annihilation methods. We prove a decomposition theorem of every chain as an ordinal sum of indecomposable $\overline{0}$-free subreducts (totally ordered semihoops) and we discuss another possible decomposition of chains in terms of connected rotation-annihilation.

4. In the fifth chapter we concentrate on the significant properties of the varieties, or equivalently of the logics, that we intend to study: three versions of standard completeness (for each of them we prove useful algebraic equivalencies), local finiteness, finite embeddability property, finite model property and decidability.

5. In the sixth chapter, we study some particular cases of the decomposition in connected rotation-annihilation, obtaining an extension of the theory of perfect, local and bipartite algebras (originally studied in the context of MV and BL-algebras) to MTL-algebras. Perfect IMTL-algebras are proved to be exactly (module isomorphism) the disconnected rotations of prelinear semihoops. This is used to prove that the lattice of varieties of bipartite IMTL-algebras is isomorphic to the lattice of varieties of prelinear semihoops. The results of this chapter are already published by the author in two joint works with F. Esteva and J. Gispert in [119, 120].

6. In the seventh chapter we study a class of indecomposable MTL-chains (w.r.t. ordinal sums), namely those chains satisfying a weak form of the cancellation law. We study their corresponding logics and their standard completeness and other algebraic properties. The results of this chapter are already published by the author in a joint work with F. Montagna and R. Horčík in [117].

7. In the eighth chapter we study the axiomatic extensions of t-norm based fuzzy logics that enjoy the global Deduction-Detachment Theorem. They are characterized by satisfying a weak form of contraction, that we call *n-contraction*. Their corresponding logics, standard completeness and algebraic properties are discussed. There is a preliminary short paper with the results of this chapter by the author in a joint work with F. Esteva and J. Gispert in [122].

8. In the ninth chapter we focus on a proper subvariety of 3-contractive MTL-algebras, the so-called *Weak Nilpotent Minimum algebras* (WNM-algebras, for short). Local finiteness is proved for all varieties of WNM-algebras, so the study reduces to the knowledge of finite chains. It enables us to give some criteria to compare varieties generated by WNM-chains. We axiomatize varieties generated by WNM-chains and discuss their standard completeness properties. A preliminar version of this results (with a mistake that is corrected in this chapter) is already published in a short paper by the author in a joint work with F. Esteva and J. Gispert in [121].

9. In the tenth chapter we move to a more expressive language by considering expansions of t-norm based logics with additional truth-constants. The motivation of such logics, which is explained in the introduction of the chapter, is mainly related to the interest in being able to exploit in a deep way the fuzziness of the logic by putting the intermediate truth-values explicitly in the language. Several distinctions in the standard completeness properties of these logics are made and then they are discussed. The results of this chapter are already published (or submitted for publication) in a series of joint papers with R. Cignoli, F. Esteva, J. Gispert, L. Godo and P. Savický in [55, 132, 50, 56].

10. Finally, an eleventh chapter collects the main results obtained in the thesis and the problems that remain open for a future research.