

**MONOGRAFIES DE L'INSTITUT D'INVESTIGACIÓ EN
INTEL·LIGÈNCIA ARTIFICIAL**



**ALGEBRAIC STUDY OF
AXIOMATIC EXTENSIONS
OF TRIANGULAR NORM
BASED FUZZY LOGICS**



Carles Noguera i Clofent

Consell Superior d'Investigacions Científiques

MONOGRAFIES DE L'INSTITUT D'INVESTIGACIÓ
EN INTEL·LIGÈNCIA ARTIFICIAL
Number 27



Institut d'Investigació
en Intel·ligència Artificial



Consell Superior
d'Investigacions Científiques

Algebraic study of axiomatic extensions of triangular norm based fuzzy logics

Carles Noguera i Clofent

Foreword by Prof. Francesc Esteva and Prof. Joan Gispert

2007 Consell Superior d'Investigacions Científiques
Institut d'Investigació en Intel·ligència Artificial
Bellaterra, Catalan Countries, Spain

Series Editor
Institut d'Investigació en Intel·ligència Artificial
Consell Superior d'Investigacions Científiques

Foreword by
Prof. Francesc Esteva i Massaguer
Institut d'Investigació en Intel·ligència Artificial
Consell Superior d'Investigacions Científiques
and
Prof. Joan Gispert i Brasó
Departament de Probabilitat, Lògica i Estadística
Universitat de Barcelona

Volume Author
Carles Noguera i Clofent
Institut d'Investigació en Intel·ligència Artificial
Consell Superior d'Investigacions Científiques



Institut d'Investigació
en Intel·ligència Artificial



Consell Superior
d'Investigacions Científiques

© 2007 by Carles Noguera i Clofent
NIPO: 653-07-042-0
ISBN: 978-84-00-08538-4
Dip. Legal: B.27922-2007

All rights reserved. No part of this book may be reproduced in any form or by any electronic or mechanical means (including photocopying, recording, or information storage and retrieval) without permission in writing from the publisher.
Ordering Information: Text orders should be addressed to the Library of the IIIA, Institut d'Investigació en Intel·ligència Artificial, Campus de la Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia, Spain.

Contents

Foreword	xi
Abstract	xiii
Acknowledgements / Agraïments	xv
1 Introduction	1
2 Universal Algebra and Algebraic Logic	11
2.1 Universal Algebra	11
2.2 Algebraic Logic	19
2.3 Some results on ordered Abelian groups	24
I MTL and its axiomatic extensions	27
3 The logic MTL	29
3.1 MTL as a substructural logic	30
3.2 MTL as an algebraizable many-valued logic	32
3.3 MTL as a t-norm based fuzzy logic	35
3.4 Axiomatic extensions of MTL	39
4 Structure of MTL-algebras	41
4.1 Basic definitions and results	41
4.2 Archimedean classes	46
4.3 The radical of MTL-algebras	47
4.4 Standard chains	49
4.5 Ordinal sums of semihoops	51
4.6 Rotation and rotation-annihilation	54
5 Properties of varieties	59
5.1 Standard completeness. Methods and results.	59
5.2 Other algebraic and logical properties	65

6	Perfect and bipartite algebras	67
6.1	Perfect and bipartite MTL-algebras	67
6.2	Local MTL-algebras	71
6.3	Perfect algebras and disconnected rotations	72
6.4	Correspondence between varieties	73
6.5	Adding the fixpoint	75
6.6	Conclusions	83
7	Weakly cancellative MTL-algebras	85
7.1	The property of weak cancellation	85
7.2	The logics of weakly cancellative chains	90
7.3	LF, FEP and FMP	93
7.4	On standard completeness theorems	98
7.5	Conclusions	101
8	n-contractive MTL-algebras	103
8.1	The n -contraction	103
8.2	n -contractive chains	105
8.3	Combining contraction and weak cancellation	109
8.4	Some properties of n -contractive fuzzy logics	110
8.5	Conclusions	112
9	Weak Nilpotent Minimum algebras	115
9.1	Varieties of NM-algebras	115
9.2	Weak nilpotent minimum algebras	118
9.3	Generic WNM-chains	124
9.4	T-norm based axiomatic extensions	127
9.5	Axiomatization of some t-norm based extensions	130
9.6	Conclusions	134
II	Partial truth in triangular norm based logics	137
10	Expansions with truth-constants	139
10.1	Adding truth-constants	142
10.2	Structure of $L_*(\mathcal{C})$ -chains	143
10.3	Completeness results	149
10.3.1	About SSC and FSSC results	149
10.3.2	About canonical standard completeness	155
10.4	Completeness results for evaluated formulae	164
10.4.1	The case of continuous t-norms	165
10.4.2	The case of WNM-fin t-norms	168
10.5	Adding truth-constants to expansions with Δ	171
10.6	Conclusions	173
11	Final conclusions and open problems	177

List of Figures

3.1	Graphic of some axiomatic extensions of MTL.	40
7.1	Graphic of axiomatic extensions of MTL obtained by adding combinations of the schemata (WC), (PC), (OS), (Div) and (Inv). All the depicted inclusions are proper.	94
8.1	Graphic of axiomatic extensions of MTL obtained by adding all combinations of the schemata (Inv), (C_n) and (WNM). All the depicted inclusions are proper.	104
9.1	Lattice of axiomatic extensions of NM.	117
9.2	An example of WNM t-norm with a finite partition.	119
9.3	Three parametric families of WNM t-norms with finite partition. . . .	120
9.4	A WNM t-norm with a finite partition such that $I_1 = \{1\}$ (left) and its corresponding t-norm on the quotient algebra $[0, 1]_*/F_a$ (right). . .	122
9.5	Example of a canonical finite WNM-chain, $\mathcal{A}_{0,3,1,2,1}^6$. Squares represent involutive elements, while circles represent the non-involutive ones. b, c, d and 1 have some associated non-involutive elements, while a (and, of course, 0) has a trivial associated interval.	124
9.6	Example of a generic chain \mathcal{A} defined by a WNM-t-norm over the real unit interval $[0, 1]$. It has a decreasing sequence $\langle a_n : n \in \omega \rangle$ of involutive elements in the negative part with a non-trivial associated interval, an increasing sequence $\langle b_n : n \in \omega \rangle$ of involutive elements in the positive part with a non-trivial associated interval, a fixpoint c with $I_c \neq \{c\}$, and $I_1 \neq \{1\}$	126
9.7	Two examples of standard WNM-chains satisfying the FPP with no involutive intervals. The chain on the left hand side has no negation point, while the chain on the right hand side has it. . .	133
9.8	An example of a standard WNM-chain satisfying the FPP with involutive intervals.	134
10.1	Diagram of expansions for $* \in \mathbf{CONT-fin} \cup \mathbf{WNM-fin}$	174

List of Tables

3.1	Some usual axiom schemata in fuzzy logics.	39
3.2	Some axiomatic extensions of MTL obtained by adding the corresponding additional axiom schemata.	39
5.1	Standard completeness properties of some axiomatic extensions of MTL.	61
5.2	Some algebraic and logical properties.	66
6.1	Standard completeness properties of fuzzy logics arising from perfect algebras.	84
7.1	Logical and algebraic properties of weakly cancellative fuzzy logics	101
8.1	Logical and algebraic properties of n -contractive fuzzy logics . .	113
10.1	(Finite) strong standard completeness results for logics of a t-norm from CONT-fin	155
10.2	(Finite) strong standard completeness results for logics of a t-norm from WNM-fin	156
10.3	Canonical standard completeness results for logics $L_*(\mathcal{C})$ when $*$ \in CONT-fin \cup WNM-fin . Recall that \otimes_c and \star_c are those WNM t-norms depicted in Figure 9.3.	164
10.4	Canonical SC and FSSC results restricted to positively evaluated formulae for logics $L_*(\mathcal{C})$ when $*$ \in CONT-fin \cup WNM-fin . . .	171

Foreword

Multivalued logical systems were studied from Łukasiewicz's early papers but they received much more attention after some infinitely-valued systems were considered as the logical systems underlying Fuzzy Logic. These logics are known as triangular norm based multivalued (residuated or fuzzy) logics because their semantics are defined over the real unit interval by a triangular norm and its residuum. The first step in this development was the definition of Product Logic, which completed the three basic continuous triangular norm based logics (with the previously studied Łukasiewicz and Gödel-Dummett logics). Later, Hájek's BL logic and, finally, MTL logic completed the framework of triangular norm based logics. From then a lot of papers have been devoted to the study of these systems and their algebraic counterparts, the corresponding varieties. Until now a lot of work has been done in the study of subvarieties of the variety of BL-algebras but not so much is studied regarding subvarieties of MTL-algebras. The first part of this monograph contains a number of deep results towards the study of these subvarieties. Even though there is not a full description of the lattice of subvarieties, there are different approaches to the problem and deep results in each of them. On the other hand and taking into account that in applications the basic notion is that of partial truth, the second part contains a new method to study multivalued logical systems obtained when adding truth constants in the language corresponding to a subalgebra of truth values (following Pavelka's approach). That part combines the interest from the applicative point of view with good and deep theoretical results.

Carles' personality, always cooperative and open minded, has been proven by the fact that the published papers are coauthored by different authors from different institutions. He has obtained the award for students of Mathematics given by the Institut d'Estudis Catalans with a monograph entitled *Lògiques borroses (Fuzzy logics)* where he described the relation between triangular norm based logics and Fuzzy Logic together with some results presented in the first part of this monograph. His solid logic and algebraic background has been decisive for the development of the work, for the collaborations and for the elaboration and readability of the monograph you have in your hands.

We hope that this work will give the readers a deep understanding of triangular norm based logics and stimulate them to a further study on the topic.

Prof. Francesc Esteva and Prof. Joan Gispert
Bellaterra and Barcelona, Catalonia, July 2007

Abstract

According to the Zadeh's famous distinction, Fuzzy Logic in narrow sense, as opposed to Fuzzy Logic in broad sense, is the study of logical systems aiming at a formalization of approximate reasoning. In the systems commonly used the strong conjunction connective is interpreted by a triangular norm (t-norm, for short) while the implication connective is interpreted by its residuum. Therefore, the usual logical systems for Fuzzy Logic are based on t-norms with a residuum. The necessary and sufficient condition for a t-norm to have a residuum is the left-continuity. In order to define the based t-norm based fuzzy logic, Esteva and Godo introduced the system MTL, which was indeed proved to be complete with respect to the semantics given by all left-continuous t-norms and their residua.

In the first part of this dissertation we have carried out an attempt to describe the axiomatic extensions of MTL, paying special attention to those which are also t-norm based. We have done it from an algebraic point of view, by exploiting the fact that these logics are algebraizable by varieties of MTL-algebras. Therefore, our study has resulted in an algebraic study of such varieties, where the final aim would be to obtain a description of the structure of their lattice and their relevant properties. Although this description has not been achieved yet, we have done several significant advances in this direction that can be classified in two groups: (a) those that spread some light over the amazing complexity of the lattice, and (b) those that describe some well-behaved parts of the lattice. More precisely:

- By considering the connected rotation-annihilation method proposed to build involutive left-continuous continuous t-norm, we have proposed a possible way to decompose MTL-chains and we have studied some particular cases of this decomposition. This has resulted in an extension of the theory of perfect, local and bipartite algebras formerly used in varieties of MV and BL-algebras, to the variety of all MTL-algebras.
- Perfect IMTL-algebras have been proved to be exactly (module isomorphism) the disconnected rotations of prelinear semihoops (a particular case of the decomposition as connected rotation-annihilation).
- The lattice of varieties generated by perfect IMTL-algebras has been proved to be isomorphic to the lattice of varieties of prelinear semihoops.

- A decomposition theorem of every MTL-chain as an ordinal sum of indecomposable prelinear semihoops has been obtained. Since all IMTL-chains are indecomposable and, as the previous item states, we have the complexity of all the lattice of varieties inside the involutive part, the description of all indecomposable prelinear semihoops seems to be a hopeless task.
- A particular class of indecomposable MTL-chains has been studied, namely weakly cancellative chains. We have studied the logics associated to these chains.
- We have studied the varieties of MTL-chains where a weak form of contraction, the so-called n -contraction law, holds. This condition yields a global form of Deduction Detachment Theorem and allows to prove several properties of their related logics.
- We have focused on a particular subvariety of 3-contractive MTL-algebras, namely Weak Nilpotent Minimum algebras, obtaining a number of results on axiomatization of their subvarieties, local finiteness, generic chains and standard completeness.

In the second part of the dissertation we consider another significant question of Fuzzy Logic: which should be the use of the intermediate truth-values? In MTL and its extensions, these truth-values for partial truth do not seem to be used in a deep way, since in the algebraization of the logics the only distinguished value is the top element. Following Pavelka's idea, we consider expansions of t -norm based logics with constants for the intermediate truth-values, allowing them to play an explicit role in the language. The originality of our proposal lies in carrying out an algebraic approach to these expansions and studying their standard completeness properties.

Acknowledgements / Agraïments:

Aquest treball no hauria estat possible sense el mestratge, la col·laboració, l'amistat o, simplement, la presència d'un munt de persones. En primer lloc, i principalment, he d'agrair als meus directors, Francesc Esteva i Massaguer i Joan Gispert i Brasó, haver-me guiat i ajudat activament en tot moment al llarg de la investigació, sempre amb paciència, immediatesa i un entusiasme contagiós; a tots els companys de l'Institut d'Investigació en Intel·ligència Artificial del CSIC, la seva acollida i l'ambient excel·lent que s'hi respira; a Lluís Godo, els innumerables consells i cops de mà; a Josep Puyol, haver-me posat a punt el programa QMORPH de José Antonio Reyes, que ens ha fornit molts exemples d'àlgebres finites; als membres del Grup de Recerca en Lògiques No-Clàssiques de Barcelona i, en general, a tots els professors del programa de doctorat en Lògica i Fonaments de la Matemàtica de la Universitat de Barcelona, haver-me ensenyat la matèria (alguns d'ells ja durant la llicenciatura) i haver-me transmès la vocació; i als companys doctorands (especialment Félix Bou, Neus Castells, Àngel García Cerdanya, David Gracia, José Gil, Rodrigo Peláez i Jip Veldman), tantes discussions enriquidores, estones d'oci, i grups de treball al llarg d'aquests anys. A Àngel García Cerdanya li he d'agrair, a més, l'atenta lectura i els comentaris a les versions preliminars de la tesi.

This work would not have been possible without several people from abroad as well. First of all, I am in debt with the coauthors of the works that are included in this dissertation: Roberto Cignoli, Rostislav Horčík, Franco Montagna and Petr Savický; professor Petr Hájek for his kind hospitality in Prague, Antonio Di Nola for the same thing in Salerno, and all the young researchers I had the pleasure to share mathematics, logic and leisure time with, during conferences and stages abroad (Libor Běhounek, Agata Ciabattoni, Cristina Coppola, Petr Cintula, Martina Danková, Daniel Donato, Tommaso Flaminio, Brunella Gerla, Rostislav Horčík, Tomáš Kroupa, Enrico Marchioni, George Metcalfe, Tiziana Pacelli, Milán Petrík and others). I also thank Petr Cintula, Rostislav Horčík and Libor Běhounek for their helpful comments to the preliminar version of the thesis.

He d'agrair el seu suport constant a la Jèssica, als meus pares Francesc i Teresa, als meus germans Aniol i Laia; a en Joan i en Jordi, l'amistat i el suport

tècnic, així com també a les bandes Fetitxe i Bandarra i a tots els amics per haver-me ajudat a trobar el contrapunt lúdic a aquests anys d'estudi.

Durant tota l'elaboració de la tesi he gaudit d'una beca FPI del ministeri espanyol d'Educació i Ciència associada als projectes de recerca LOGFAC TICC2001-1577-C03-01 i MULOG TIN2004-07933-C03-01.

This thesis has been written by using only free software: Kubuntu GNU/Linux as operative system, Kile as text editor and pdfTeX as L^AT_EX compiler.

*A la Jèssica,
tot esperant que algun dia
la nostra petita pàtria
esdevingui lliure i completa*

Chapter 1

Introduction

Since it was founded by Aristotle, Logic has been the science devoted to the laws of the correct reasoning. Traditionally, one of the fundamental laws has been the Bivalence Principle, which states that every proposition is either true or false, independently how difficult might be in some cases to determine its truth value. This traditional logic under the Bivalence Principle, that we call *Classical Logic*, turned out to be an excellent tool for the mathematical work, specially after Mathematical Logic was born in the nineteenth century with Augustus de Morgan, George Boole and Gottlob Frege among others. It is not strange, since Mathematics do use precise concepts and always works with statements that are intended to be either true or false.

Nevertheless, Aristotle already noticed that many of the concepts that are commonly used outside the strict mathematical discourse are far from being precise; on the contrary they refer to qualities that admit degrees. In *Categories* 8, 10b 26-32 he writes:

Qualifications admit of a more and a less; for one thing is called more pale or less pale than another, and more just than another. Moreover, it itself sustains increase (for what is pale can still become paler) – not in all cases though, but in most. It might be questioned whether one justice is called more a justice than another, and similarly for the other conditions.¹

Some lines below (*Categories* 8, 11a 2-5) he adds:

At any rate things spoken of in virtue of these unquestionably admit of a more and a less: one man is called more grammatical than another, juster, healthier, and so on. Triangle and square do not seem to admit of a more, nor does any other shape.

This kind of predicates appear in propositions that often do not seem neither completely true nor completely false. This is the vagueness phenomenon and it becomes a real challenge for Logic when one considers some reasonings that involve vague predicates, such as the so called *Sorites Paradox*:

Premises:

- (1) A man who has twenty thousand hairs on his head is not bald.

¹We cite from the English translation in [6].

- (2) If a man who is not bald loses one hair, he is still not bald.

Conclusion:

- (3) A man with no hair on his head is not bald.

This reasoning is a paradox because it seems to be correct (one can derive the conclusion from the premisses by using Modus Ponens twenty thousand times), and it derives a clearly false conclusion from (apparently) true premisses (we have no doubt that (1) is true, and (2) seems also true); something that can never happen in a correct reasoning.

Several solutions to cope with the vagueness phenomena have been proposed (see e.g. [139]). One of them is Fuzzy Logic. As other non-classical logics, it rejects the Bivalence Principle and proposes infinitely-valued logics to substitute Classical Logic. Historically this idea comes from the theory of Fuzzy Sets proposed in 1965 by Lotfi Zadeh in [140]. His idea consisted on modeling vague predicates with fuzzy sets, i.e. sets where the objects can belong in a greater or lesser degree. Formally, a fuzzy set is a pair $\langle X, \mu \rangle$ where X is a set (in the classical sense) and $\mu : X \rightarrow [0, 1]$ is a function (called *membership function*) that maps every object $x \in X$ to a real number $\mu(x)$ between 0 and 1, that is interpreted as the degree of membership of the object in the fuzzy set. For instance, given the vague predicate *tall*, one can define a fuzzy set by considering: $X := [0.3, 2.4]$ (set of possible heights in meters) and the membership function:

$$\mu(x) = \begin{cases} 0 & \text{if } x < 1.2, \\ \frac{5}{3}x - 2 & \text{if } 1.2 \leq x \leq 1.8, \\ 1 & \text{if } x > 1.8. \end{cases}$$

This fuzzy set models the predicate *tall* in such a way that people whose height is more than 1.8 meters are definitely tall, people with less than 1.2 meters of height are definitely not tall and people with an intermediate height is given the quality *tall* in an intermediate degree according to a linear function.² This interpretation amounts to a certain use of infinitely-many truth-values in the following way: if some individual a in our universe of discourse has a height $x \in X$, then we say that the sentence ' a is tall' is true at degree $\mu(x)$.

However, if one wants to model vague predicates in terms of fuzzy sets, some way to combine them is required, since one will want to consider propositions where vague predicates are combined in disjunctions, conjunctions, negations and other usual logical connectives. For instance, to model in a truth-functional way the conjunction of two vague predicates one can consider their corresponding fuzzy sets and provide some binary function such that, given any object and its membership degrees to the fuzzy sets, returns the membership degree to the intersection. In the same way, a binary function is needed for the union of fuzzy sets, and a unary function for the complement. In his foundational papers

²Of course, the choice of the membership function must depend on the context in which one wants to model the vague predicate. For instance, the meaning of *tall* is not the same when the universe of individuals are basketball players or just ordinary people.

Zadeh proposed the functions $\min\{x, y\}$, $\max\{x, y\}$ and $1 - x$ respectively for the intersection, the union and the complement of fuzzy sets.

In 1969 Goguen proposes in [75] a solution to the Sorites Paradox in terms of fuzzy sets, but using some other functions to combine them, namely the truth-functions introduced by Łukasiewicz for some infinitely-valued logics. Therefore, a few words on the origin of many-valued logics are needed here.

The first many-valued logic was introduced in 1918 by Jan Łukasiewicz. It was a three-valued logic proposed to deal with the problem of future contingents. According to Łukasiewicz, the Bivalence Principle implies a kind of determinism, since it forces all propositions to be either true or false, including those that state some facts about the future.³ He believes that it is more intuitive to claim that these propositions are still neither true nor false, and thus he introduces a new truth-value for them that he calls *possible*. If $\{0, \frac{1}{2}, 1\}$ is the set formed by his three truth-values, Łukasiewicz defines the logical connectives of implication and negation by means of the following tables:

\rightarrow	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

	\neg
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

Therefore, he is using the Principle of Extensionality to compute the truth-value of every complex proposition from the truth-values of its parts by means of the tables. The meaning of the tables can be almost explained in the following way. Suppose that $\{T, F\}$ are the classical truth-values, *true* and *false*. Then, the three Łukasiewicz's truth-values can be interpreted as sets of these classical truth-values: $0 = \{F\}$, $1 = \{T\}$ and $\frac{1}{2} = \{T, F\}$, since the value $\frac{1}{2}$ is given to those proposition about future facts which still we do not know whether they will be true or false. Almost all the values in the tables are obtained by operating all the elements in these sets according to the classical connectives of implication and negation. For instance, $\{T, F\} \rightarrow \{T\} = \{T\}$, because according to the classical implication we have $T \rightarrow T = T$ and $F \rightarrow T = T$. There is only one exception which is not compatible with this interpretation: $\frac{1}{2} \rightarrow \frac{1}{2}$ should be $\frac{1}{2}$, but it is 1. The obvious reason seems to be that Łukasiewicz wanted to preserve the validity of the Identity Law, $\varphi \rightarrow \varphi$.

In 1922 Łukasiewicz generalizes the three-valued logic to an n -valued logic for every $n \geq 4$ finite, where the set of truth-values is $\{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$, and the logical operations are defined by $x \rightarrow y := \min\{1, 1 - x + y\}$ and $\neg x := 1 - x$.

³The problem of future contingents can be also traced back to Aristotle; viz. his famous naval battle example.

Finally, in 1930, with Alfred Tarski in [110], they generalize it to an infinitely-valued logic where the set of truth-values is the real unit interval and the truth-functions are defined as in the finitely-valued case. Several additional connectives are defined in the following way: $x \& y := \neg(x \rightarrow \neg y)$, $x \vee y := (x \rightarrow y) \rightarrow y$ and $x \wedge y := \neg(\neg x \vee \neg y)$. Then, some properties of the classical conjunction split between $\&$ and \wedge :

1. For every a, b, c , $a \& b \leq c$ iff $a \leq b \rightarrow c$ (residuation law)
2. For every a, b , $a \rightarrow b = 1$ iff $a \wedge b = a$ iff $a \leq b$ ($\wedge = \min$)

By using these truth-functions, Goguen was able to propose a solution to the sorites paradox. Indeed, he considered that *bald* is a vague predicate and thus it must be interpreted by a fuzzy set. (1) can be given the minimum truth-value, since a man with twenty thousand hairs on his head is not bald, i.e. he belongs to the set of bald men at degree 0. However, (2) is not absolutely true. Consider that its truth-value is $\frac{19999}{20000}$, almost 1. Let v_i be the truth-value of 'A man with i hairs in his head is not bald'. Then, we have $v_{20000} = 1$ and $v_i \rightarrow v_{i-1} = \frac{19999}{20000}$. Interpreting \rightarrow with Łukasiewicz implication and after an easy computation, it comes out that $v_0 = 0$, hence the paradox disappears.

Afterwards, following the truth-functional setting of Zadeh, some other functions were proposed to model the combination of fuzzy sets. They were required to satisfy certain conditions. For instance, the functions for disjunction and conjunction were required to be associative and commutative. Alsina, Trillas and Valverde (see e.g. [4]) proposed a class of functions taken from the theory of probabilistic metric spaces (see [133, 134]), the triangular norms (t-norms, for short), to model the intersection of fuzzy sets, their dual functions, the t-conorms for the unions, and the so-called weak negation functions for the complement (see [136, 48]). T-norms are binary operations defined on the real unit interval which are associative, commutative, monotonic and have 1 as neutral element. As regards to implication, mainly two kinds were proposed:

S-implications: those satisfying that for every a, b , $a \rightarrow b = \neg a \vee b$, and
 R-implications: those satisfying the residuation law.

In [137] R-implications were chosen in order to deal with the Modus Ponens rule. For every continuous t-norm there is an associated R-implication, which is called its *residuum*. Although the continuity was not required in the definition of t-norm, the majority of the known examples were actually continuous, for instance the minimum (the original interpretation for the intersection of fuzzy sets in Zadeh's seminal paper), the Łukasiewicz interpretation of the connective $\&$ (we will call it *Łukasiewicz t-norm*) and the product of reals (we will call it *product t-norm*). Moreover, all continuous t-norms can be represented as an ordinal sum of these three examples, as proved in [118] and in [108].

Interestingly, two of the three basic continuous t-norms and their residua had been used in the semantics of some infinitely-valued logics before fuzzy sets were defined. On the one hand, the Łukasiewicz t-norm, as we have already explained, appeared in the semantics of Łukasiewicz's infinitely-valued logic. He also defined a Hilbert-style calculus whose axioms were:

- (L1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
- (L2) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
- (L3) $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$
- (L4) $(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi)$
- (L5) $((\varphi \rightarrow \psi) \rightarrow (\psi \rightarrow \varphi)) \rightarrow (\psi \rightarrow \varphi)$

and Modus Ponens was the only inference rule. Let \mathcal{A} be the algebra defined over $[0, 1]$ by the Lukasiewicz truth-functions. He conjectured that the tautologies of the infinitely-valued logic given by the matrix $\langle \mathcal{A}, \{1\} \rangle$ coincide with the theorems of this Hilbert-style calculus. However, he was not able to prove it. In 1935 Mordchaj Wajsberg claimed that he had found a proof, but he never gave it. The conjecture was finally proved by syntactical means in 1958 by Rose and Rosser [131] and algebraically in 1959 by Chang [25, 26]. It is important to remark that Meredith showed in [113] the redundancy of the axiom (L5) and that Hay improved the result in 1963 (in [86]) by proving that the Hilbert-style calculus in fact coincides with the finitary fragment of Lukasiewicz logic.

On the other hand, the minimum t-norm also corresponds to the semantical interpretation of the conjunction in a many-valued logic. Indeed, Gödel used it in [74] to study some linearly ordered matrix semantics for superintuitionistic logics. Dummett gave in [47] a sound and complete Hilbert-style calculus for the infinitely-valued logic corresponding to the matrix defined over $[0, 1]$ by the minimum t-norm and its residuated implication.

These two examples suggested that, with the usage of t-norms and their residua, Fuzzy Logic was very close to the apparently independent field of many-valued logics. The next step in approaching both fields was done in [83] when Hájek, Godó and Esteva gave also a Hilbert-style calculus for the remaining prominent example of continuous t-norm: the product t-norm.

Therefore, it became clear that at least some part of Fuzzy Logic was directly related to the study of some many-valued logics. This led to the distinction by the founder of the field, Zadeh, between a wide and a narrow sense of 'Fuzzy Logic'. In [141] he writes:

The term 'Fuzzy Logic' has two different meanings: wide and narrow. In a narrow sense, fuzzy logic, FLn, is a logical system which aims at a formalization of approximate reasoning. In this sense, FLn is an extension of multivalued logic. However, the agenda of FLn is quite different from that of traditional multivalued logics. In particular, such key concepts in FLn as the concept of a linguistic variable, canonical form, fuzzy if-then rule, fuzzy quantification and defuzzification, predicate modification, truth qualification, the extension principle, the compositional rule of inference and interpolative reasoning, among others, are not addressed in traditional systems. This is the reason why FLn has a much wider range of applications than traditional systems. In its wide sense, fuzzy logic, FLw, is fuzzily synonymous with the fuzzy set theory, FST, which is the theory of classes with unsharp boundaries. FST is much broader than FLn and includes the latter as one of its branches.

Once it was shown that the logics defined by the three main continuous t-norms and their residua enjoyed a syntactical calculus, Hájek proposed in [79] a

new logical system, that he called Basic Fuzzy Logic (BL, for short) to capture the logic given by the class of all continuous t-norms.⁴ His conjecture was proved in [80, 30]. Nevertheless, Esteva and Godo noticed that the necessary and sufficient condition for a t-norm to have a residuum was not the continuity, but the left-continuity. Thus, they wanted to find the fuzzy logic corresponding to the bigger class of all left-continuous t-norms. To fulfil this aim, they proposed a new Hilbert-style calculus in [51] called Monoidal T-norm based Logic (MTL, for short). MTL was proved to be indeed the logic of all left-continuous t-norms and their residua by Jenei and Montagna in [100]. Thus, in a sense MTL can be considered the real basic fuzzy logic, since it is the weakest logic which is complete with respect to a semantics given by a class of t-norms and their residua (this kind of logics are called *t-norm based fuzzy logics* or just *t-norm based logics*). The word 'Monoidal' in the name is due to the fact that MTL is an axiomatic extension of another many-valued logic proposed by H  hle in [87] called *Monoidal Logic* (ML, for short). ML has been proved to be equivalent to a contraction-less substructural logic, namely the system H_{BCK} (also called FL_{ew}) of Ono and Komori (see [127, 126]). Therefore, since MTL enjoys neither the contraction rule, it can be regarded not only as a fuzzy logic and as a many-valued logic, but also as a substructural logic.

In this dissertation, that pertains to Fuzzy Logic in narrow sense, we study MTL (and its axiomatic extensions) from the two first points of view of the previous list: as a fuzzy logic and as a many-valued logic. More than that, we study MTL as an *algebraizable* many-valued logic. Indeed, traditionally many algebraic counterparts have been used in the research on many-valued logics. For instance, Chang introduced MV-algebras to study Łukasiewicz logic⁵, a subclass of Heyting algebras called *G-algebras* have been introduced for G  del-Dummett logic, residuated lattices (in the sense of [43]) for ML and so on. H  jek gave an algebraic semantics for BL, namely the variety of BL-algebras, a subclass of residuated lattices. A bigger variety of residuated lattices has been used to algebraize MTL by Esteva and Godo, the variety of MTL-algebras \mathbf{MTL} . Actually, MTL is algebraizable in the sense of Blok and Pigozzi (see [19]) and \mathbf{MTL} is its equivalent quasivariety semantics. Therefore, all its finitary extensions are also algebraizable and their algebraic semantics are the corresponding subquasivarieties of \mathbf{MTL} . When we restrict to axiomatic extensions, the corresponding algebraic semantics are the subvarieties of \mathbf{MTL} . Moreover, the algebraizability implies a number of results connecting algebraic properties of the semantics with logical properties, the so-called *bridge theorems* of Abstract Algebraic Logic (see [62]). A relevant subclass of MTL-algebras are those defined over the real unit interval, which are exactly those where the conjunction $\&$ is interpreted by a left-continuous t-norm. These algebras are called *standard*. For every finitary extension of MTL a crucial question, from the Fuzzy Logic point of view, is whether it is complete w.r.t. the standard algebras of its corresponding quasi-

⁴Łukasiewicz, G  del-Dummett and Product logics were proved to be axiomatic extensions of BL.

⁵A polynomially equivalent algebraic semantics for Łukasiewicz logic, the class of Wajsberg algebras, has been proposed in [130, 63].

variety. This kind of result is called *standard completeness*.⁶

Many investigations in the algebraic direction for MTL and its finitary extensions have been already carried out (see e.g. [30, 100, 49, 53, 3, 32, 33, 81, 71, 72, 116, 88, 89, 90, 69, 91]). Thanks to these works (and others) some parts of the lattice of subvarieties of MTL are already known, namely all varieties of MV-algebras (see [103]), all varieties of G-algebras (see e.g. [76]), many varieties of BL-algebras (see [3, 53]) and some other parts of the lattice (see e.g. [71, 72]). The results obtained in BL strongly relied on the knowledge on its linearly ordered algebras, since all BL-algebras (in fact, all MTL-algebras) are representable as subdirect product of linearly ordered ones, and linearly ordered BL-algebras were well described in terms of ordinal sums of some basic components (generalizing the corresponding result for continuous t-norms). Unfortunately, such a representation is not known for linearly ordered MTL-algebras (also called *MTL-chains*). Thus, the structure of the lattice of varieties of MTL-algebras was very far from being known.

With this dissertation we want to contribute to the task of describing axiomatic extensions of t-norm based fuzzy logics, or equivalently, varieties of MTL-algebras and their properties. On the one hand, we try to describe the structure of MTL-chains. Our first attempt consists in using the decomposition as ordinal sum of indecomposable semihoops. However, although we prove that for every chain exists a maximum decomposition in this sense, we show that the class of indecomposable semihoops is really huge and it seems too difficult to describe. Nevertheless, we study a significant class of indecomposable semihoops: those satisfying a weak form of cancellation. The second attempt uses one of the methods proposed by Jenei to built left-continuous t-norms, the so-called connected rotation-annihilation construction. We show that for every chain there is also a maximal decomposition as connected rotation-annihilation. The task of describing the indecomposable chains seems also too far away, but we can still study some cases of this decomposition, which leads to the theory of perfect and bipartite MTL-algebras. On the other hand, we focus on several varieties of MTL-algebras satisfying some nice properties, namely the varieties of n -contractive algebras. They seem a good choice because they correspond to the axiomatic extensions of MTL satisfying the global Deduction-Detachment Theorem, and they contain all the locally finite varieties of MTL-algebras, in particular the weak nilpotent minimum algebras (a subvariety that we study in a greater detail). For all the varieties of MTL-algebras defined in the thesis we study several relevant logical and algebraic properties, mainly standard completeness properties, local finiteness, finite embedding property, finite model property and decidability.

⁶The logics studied in this thesis belong to another interesting class of logics studied by general methods, namely the class of Weakly Implicative fuzzy logics (see [38]). This class is claimed to be the right class of fuzzy logics (in narrow sense) in [12]. Using the general results mentioned above the authors obtain completeness w.r.t. linearly ordered algebras, local Deduction-Detachment Theorem, subdirect decomposition theorem, etc. Moreover, we also would like to point out that axiomatic extensions of MTL are *core fuzzy logics* in the sense of [82].

Finally, we consider another aspect of Fuzzy Logic in narrow sense: which should be the use of the intermediate truth-values? In MTL and its finitary extensions, these truth-values do not seem to be used in a deep way, since in their algebraization the only distinguished value is the top element, as in classical logic. For this reason, following Pavelka's idea in [128] for Łukasiewicz logic, we consider expansions of t-norm based logics with constants for the intermediate truth-values, allowing them to play an explicit role in the language. The originality of our proposal lies in carrying out an algebraic approach to these expansions, i.e. studying them also as algebraizable logics.

The thesis is structured in eleven chapters. After this introduction, a first group of chapters (from the second till the fifth) are introductory, with necessary preliminaries, sometimes interpreted from our point of view; the rest contains our contributions to the field.

1. In the second chapter we introduce the basic notions and notation that will be used throughout all the dissertation on Universal Algebra and Algebraic Logic.
2. In the third chapter the logic MTL, the main subject of the study, is formally introduced. The three perspectives (as a substructural logic, as an algebraizable many-valued logic and as a t-norm based fuzzy logic) for MTL are also formally discussed.
3. In the fourth chapter we present the basic results and definitions for MTL-algebras that we will need to study their varieties. In particular, the decomposition of every algebra as a sudirect product of chains is proved and some useful methods to build new IMTL-algebras are introduced, namely Jenei's rotation and rotation-annihilation methods. We prove a decomposition theorem of every chain as an ordinal sum of indecomposable $\bar{0}$ -free subreducts (totally ordered semihoops) and we discuss another possible decomposition of chains in terms of connected rotation-annihilation.
4. In the fifth chapter we concentrate on the significant properties of the varieties, or equivalently of the logics, that we intend to study: three versions of standard completeness (for each of them we prove useful algebraic equivalencies), local finiteness, finite embeddability property, finite model property and decidability.
5. In the sixth chapter, we study some particular cases of the decomposition in connected rotation-annihilation, obtaining an extension of the theory of perfect, local and bipartite algebras (originally studied in the context of MV and BL-algebras) to MTL-algebras. Perfect IMTL-algebras are proved to be exactly (module isomorphism) the disconnected rotations of prelinear semihoops. This is used to prove that the lattice of varieties of bipartite IMTL-algebras is isomorphic to the lattice of varieties of prelinear semihoops. The results of this chapter are already published by the author in two joint works with F. Esteva and J. Gispert in [119, 120].

6. In the seventh chapter we study a class of indecomposable MTL-chains (w.r.t. ordinal sums), namely those chains satisfying a weak form of the cancellation law. We study their corresponding logics and their standard completeness and other algebraic properties. The results of this chapter are already published by the author in a joint work with F. Montagna and R. Horčík in [117].
7. In the eighth chapter we study the axiomatic extensions of t-norm based fuzzy logics that enjoy the global Deduction-Detachment Theorem. They are characterized by satisfying a weak form of contraction, that we call *n-contraction*. Their corresponding logics, standard completeness and algebraic properties are discussed. There is a preliminary short paper with the results of this chapter by the author in a joint work with F. Esteva and J. Gispert in [122].
8. In the ninth chapter we focus on a proper subvariety of 3-contractive MTL-algebras, the so-called *Weak Nilpotent Minimum algebras* (WNM-algebras, for short). Local finiteness is proved for all varieties of WNM-algebras, so the study reduces to the knowledge of finite chains. It enables us to give some criteria to compare varieties generated by WNM-chains. We axiomatize varieties generated by WNM-chains and discuss their standard completeness properties. A preliminar version of this results (with a mistake that is corrected in this chapter) is already published in a short paper by the author in a joint work with F. Esteva and J. Gispert in [121].
9. In the tenth chapter we move to a more expressive language by considering expansions of t-norm based logics with additional truth-constants. The motivation of such logics, which is explained in the introduction of the chapter, is mainly related to the interest in being able to exploit in a deep way the fuzziness of the logic by putting the intermediate truth-values explicitly in the language. Several distinctions in the standard completeness properties of these logics are made and then they are discussed. The results of this chapter are already published (or submitted for publication) in a series of joint papers with R. Cignoli, F. Esteva, J. Gispert, L. Godo and P. Savický in [55, 132, 50, 56].
10. Finally, an eleventh chapter collects the main results obtained in the thesis and the problems that remain open for a future research.

Chapter 2

Universal Algebra and Algebraic Logic preliminaries

In this chapter we introduce the basic definitions and notation that will be used throughout the dissertation. Since this is an algebraic investigation of a certain family of logics, it is necessary to introduce both several concepts from two fields: Universal Algebra and Algebraic Logic.

2.1 Universal Algebra

As regards to Universal Algebra, it will be for us a language and a tool, rather than a topic of research. Therefore, instead of doing an extensive presentation of its main contents, we will just assume some acquaintance of the reader with the topic (a pair of good reference textbooks are [24] and [78]) and set the main definitions and results, and the notation as it will be used in the dissertation.

An *algebraic language* is a pair $\mathcal{L} = \langle F, \tau \rangle$, where F is a set of functional symbols and τ is a mapping $\tau : F \rightarrow \omega$ (where ω denotes the set of the natural numbers). For every $f \in F$, $\tau(f)$ is called the *arity* of the functional symbol f .

Example 1. *As an example we may cite the main language that will appear all along the dissertation. Consider the pair $\mathcal{L} = \langle \{\&, \rightarrow, \wedge, \vee, \bar{0}, \bar{1}\}, \tau \rangle$, where $\tau(\&) = \tau(\rightarrow) = \tau(\wedge) = \tau(\vee) = 2$ and $\tau(\bar{0}) = \tau(\bar{1}) = 0$. This definition will be sometimes simplified just by saying that $\mathcal{L} = \{\&, \rightarrow, \wedge, \vee, \bar{0}, \bar{1}\}$ is a language of type $\langle 2, 2, 2, 2, 0, 0 \rangle$.*

Given an algebraic language $\mathcal{L} = \langle F, \tau \rangle$, an algebra \mathcal{A} of type \mathcal{L} is a pair $\langle A, \{f^{\mathcal{A}} : f \in F\} \rangle$, where A is a non-empty set called the *universe* or the *carrier* of \mathcal{A} and for every $f \in F$, if $\tau(f) = n$, then $f^{\mathcal{A}}$ is an n -ary operation in A (a 0-ary operation in A is just an element of A). When the number of functionals

is finite, say $\{f_1, \dots, f_n\}$, we write $\mathcal{A} = \langle A, f_1^{\mathcal{A}}, \dots, f_n^{\mathcal{A}} \rangle$, and we say that it is an algebra of type $\langle \tau(f_1), \dots, \tau(f_n) \rangle$. The superscripts in the functions will be often omitted when they are clear from the context.

Example 2. An important example of algebra is the algebra of formulae. Let \mathcal{L} be a countable algebraic language and let X be an infinite countable set. The set $Fm_{\mathcal{L}}(X)$ of \mathcal{L} -formulae (or \mathcal{L} -terms) over X is inductively defined as:

1. For every $x \in X$, $x \in Fm_{\mathcal{L}}(X)$.
2. For every $c \in F$ with arity 0, $c \in Fm_{\mathcal{L}}(X)$.
3. For every $f \in F$ with arity $n > 0$, if $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}(X)$, then $f(\varphi_1, \dots, \varphi_n) \in Fm_{\mathcal{L}}(X)$.

X is called the set of variables. The algebra of formulae, $\mathbf{Fm}_{\mathcal{L}}(X) = \langle Fm_{\mathcal{L}}(X), \{f^{\mathbf{Fm}_{\mathcal{L}}(X)} : f \in F\} \rangle$, is defined by:

- For every $c \in F$ with arity 0, $c^{\mathbf{Fm}_{\mathcal{L}}(X)} := c$.
- For every $f \in F$ with arity $n > 0$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}(X)$, $f^{\mathbf{Fm}_{\mathcal{L}}(X)}(\varphi_1, \dots, \varphi_n) := f(\varphi_1, \dots, \varphi_n)$.

Given another infinite countable set Y of variables, the resulting algebra of formulae, $\mathbf{Fm}_{\mathcal{L}}(Y)$ is isomorphic to $\mathbf{Fm}_{\mathcal{L}}(X)$, thus to simplify the notation the set of formulae will be denoted as $Fm_{\mathcal{L}}$. Notice that $Fm_{\mathcal{L}}$ is also countable.

The algebra of formulae is defined in the same way when the set of variables is uncountable, but since the number of variables occurring in a formula is always finite, in general it is enough to consider algebras of formulae built over an infinite countable set of variables.

We write $\varphi(x_1, \dots, x_n)$ to indicate that the variables occurring in the formula φ are among $\{x_1, \dots, x_n\}$.

Example 3. Let $\mathcal{A} = \langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ be an algebra of type $\langle 2, 2, 0, 0 \rangle$. It is a bounded lattice if, and only if, the operations $\wedge^{\mathcal{A}}$ and $\vee^{\mathcal{A}}$ are associative, commutative and idempotent, and for every $a, b \in A$ it holds:

- $a \wedge^{\mathcal{A}} \bar{0}^{\mathcal{A}} = \bar{0}^{\mathcal{A}}$
- $a \vee^{\mathcal{A}} \bar{1}^{\mathcal{A}} = \bar{1}^{\mathcal{A}}$
- $a \wedge^{\mathcal{A}} (a \vee^{\mathcal{A}} b) = a$
- $a \vee^{\mathcal{A}} (a \wedge^{\mathcal{A}} b) = a$

\mathcal{A} is a distributive lattice iff it is a lattice such that for every $a, b, c \in A$ it holds:

- $a \wedge^{\mathcal{A}} (b \vee^{\mathcal{A}} c) = (a \wedge^{\mathcal{A}} b) \vee^{\mathcal{A}} (a \wedge^{\mathcal{A}} c)$

In every lattice \mathcal{A} it is possible to define a partial order in the following way: for every $a, b \in A$, $a \leq b$ iff $a \wedge^{\mathcal{A}} b = a$ (or, equivalently, $a \vee^{\mathcal{A}} b = b$). In this partial order every set of two elements $\{a, b\}$ has an infimum (namely $a \wedge^{\mathcal{A}} b$) and a supremum (namely $a \vee^{\mathcal{A}} b$). A lattice is called complete if, and only if, every subset of the carrier (even the infinite ones) has supremum and infimum.

Let $\mathcal{A} = \langle A, \{f^{\mathcal{A}} : f \in F\} \rangle$ and $\mathcal{B} = \langle B, \{f^{\mathcal{B}} : f \in F\} \rangle$ be two algebras of the same type. We say that \mathcal{A} is a subalgebra of \mathcal{B} , and we write $\mathcal{A} \subseteq \mathcal{B}$, if and only if:

- $A \subseteq B$,
- for every $c \in F$ with arity 0, $c^{\mathcal{A}} = c^{\mathcal{B}}$, and
- for every $f \in F$ with arity $n > 0$, $f^{\mathcal{A}} = f^{\mathcal{B}} \upharpoonright A^n$.

The universe of a subalgebra of \mathcal{A} is called a *subuniverse*. Since the set of subuniverses of \mathcal{A} is closed under arbitrary intersections, for every non-empty $B \subseteq A$, one can define the subalgebra generated by B as the subalgebra $\langle B \rangle_{\mathcal{A}}$ whose universe is $\bigcap \{C \subseteq A : C \text{ is a subuniverse of } \mathcal{A} \text{ and } B \subseteq C\}$.

A mapping $h : A \rightarrow B$ is a *homomorphism* from \mathcal{A} to \mathcal{B} if and only if:

- for every $c \in F$ with arity 0, $h(c^{\mathcal{A}}) = c^{\mathcal{B}}$, and
- for every $f \in F$ with arity $n > 0$ and for every $a_1, \dots, a_n \in A$, $h(f^{\mathcal{A}}(a_1, \dots, a_n)) = f^{\mathcal{B}}(h(a_1), \dots, h(a_n))$.

\mathcal{B} is *homomorphic image* of \mathcal{A} if, and only if, there is a surjective homomorphism from \mathcal{A} to \mathcal{B} . A one-to-one homomorphism is called an *embedding* and a one-to-one surjective homomorphism is called an *isomorphism*. We say that \mathcal{A} and \mathcal{B} are *isomorphic* and we write $\mathcal{A} \cong \mathcal{B}$ if, and only if, there is an isomorphism from \mathcal{A} to \mathcal{B} .

Given an algebra $\mathcal{A} = \langle A, \{f^{\mathcal{A}} : f \in F\} \rangle$, a set $\theta \subseteq A^2$ is a *congruence* of \mathcal{A} if, and only if, it is an equivalence relation on A and for every $f \in F$ with arity $n > 0$, if $\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle \in \theta$, then $\langle f^{\mathcal{A}}(a_1, \dots, a_n), f^{\mathcal{A}}(b_1, \dots, b_n) \rangle \in \theta$. The set of all congruences of \mathcal{A} is denoted as $Con(\mathcal{A})$ and it is closed under arbitrary intersections, hence it forms a bounded complete lattice ordered by the inclusion: $\mathbf{Con}(\mathcal{A}) = \langle Con(\mathcal{A}), \cap, \bigvee, \Delta_{\mathcal{A}}, \nabla_{\mathcal{A}} \rangle$, where $\bigvee_{i \in I} \theta_i = \bigcap \{ \theta : \bigcup_{i \in I} \theta_i \subseteq \theta \}$, $\Delta_{\mathcal{A}} = \{ \langle a, a \rangle : a \in A \}$ and $\nabla_{\mathcal{A}} = A^2$. Since this is a complete lattice, it makes sense to consider the notion of *generated congruence*, i.e. given $B \subseteq A^2$ there exists the minimum congruence containing B , which is denoted as $\Theta(B)$. The congruences of the form $\Theta(\{ \langle a, b \rangle \})$, are called *principal congruences*. Given $\theta_1, \theta_2 \in Con(\mathcal{A})$, the composition of θ_1 with θ_2 is defined as the binary relation $\theta_1 \circ \theta_2 := \{ \langle a, b \rangle : \text{there is } c \in A \text{ such that } \langle a, c \rangle \in \theta_1 \text{ and } \langle c, b \rangle \in \theta_2 \}$. We say that \mathcal{A} is *simple* if, and only if, $Con(\mathcal{A}) = \{ \Delta_{\mathcal{A}}, \nabla_{\mathcal{A}} \}$.

Let $\mathcal{A} = \langle A, \{f^{\mathcal{A}} : f \in F\} \rangle$ be an algebra and $\theta \in Con(\mathcal{A})$. Given $a \in A$, its equivalence class with respect to θ is denote as a/θ . The *quotient algebra* of \mathcal{A} by θ is defined as $\mathcal{A}/\theta = \langle A/\theta, \{f^{A/\theta} : f \in F\} \rangle$ where:

- $A/\theta = \{a/\theta : a \in \theta\}$,
- for every $c \in F$ with arity 0, $c^{A/\theta} = c^A/\theta$, and
- for every $f \in F$ with arity $n > 0$ and for every $a_1, \dots, a_n \in A$, $f^{A/\theta}(a_1/\theta, \dots, a_n/\theta) = f^A(a_1, \dots, a_n)/\theta$.

Given a family $\{\mathcal{A}_i : i \in I\}$ of algebras of the same type, we define the *product algebra* (or *direct product algebra*) $\prod_{i \in I} \mathcal{A}_i = \langle \prod_{i \in I} A_i, \{f^{\prod_{i \in I} \mathcal{A}_i} : f \in F\} \rangle$ by:

- $\prod_{i \in I} A_i$ is the Cartesian product of the universes,
- for every $c \in F$ with arity 0, $c^{\prod_{i \in I} \mathcal{A}_i} = \langle c^{\mathcal{A}_i} : i \in I \rangle$, and
- for every $f \in F$ with arity $n > 0$ and for every $\hat{a}_1, \dots, \hat{a}_n \in \prod_{i \in I} A_i$, $f^{\prod_{i \in I} \mathcal{A}_i}(\hat{a}_1, \dots, \hat{a}_n)(i) = f^{\mathcal{A}_i}(\hat{a}_1(i), \dots, \hat{a}_n(i))$, for every $i \in I$, where $\hat{b}(i)$ denotes the i -th component of $\hat{b} \in \prod_{i \in I} A_i$.

If $j \in I$, the j -th *projection* is the homomorphism $\pi_j : \prod_{i \in I} \mathcal{A}_i \rightarrow \mathcal{A}_j$ defined as $\pi(\hat{a}) := \hat{a}(j)$.

A filter \mathcal{F} on a set I is a family of subsets of I such that:

- $I \in \mathcal{F}$,
- if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$, and
- if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathcal{F}$.

\mathcal{F} is proper if, and only if, $\emptyset \notin \mathcal{F}$ (i.e. $\mathcal{F} \neq \mathcal{P}(I)$).

Given a family $\{\mathcal{A}_i : i \in I\}$ of algebras of the same type and a proper filter \mathcal{F} on I , the following binary relation is defined on $\prod_{i \in I} A_i$: for every $\hat{a}, \hat{b} \in \prod_{i \in I} A_i$, $\hat{a} \sim_{\mathcal{F}} \hat{b}$ if, and only if, $\{i \in I : \hat{a}(i) = \hat{b}(i)\} \in \mathcal{F}$. $\sim_{\mathcal{F}}$ is a congruence of $\prod_{i \in I} \mathcal{A}_i$. The *reduced product algebra* of $\{\mathcal{A}_i : i \in I\}$ w.r.t. \mathcal{F} is the algebra $\prod_{i \in I} \mathcal{A}_i / \mathcal{F} = \langle \prod_{i \in I} A_i / \mathcal{F}, \{f^{\prod_{i \in I} \mathcal{A}_i / \mathcal{F}} : f \in F\} \rangle$ defined by:

- $\prod_{i \in I} A_i / \mathcal{F}$ is the quotient by $\sim_{\mathcal{F}}$ of the Cartesian product $\prod_{i \in I} A_i$,
- for every $c \in F$ with arity 0, $c^{\prod_{i \in I} \mathcal{A}_i / \mathcal{F}} = c^{\prod_{i \in I} \mathcal{A}_i} / \mathcal{F}$, and
- for every $f \in F$ with arity $n > 0$ and for every $\hat{a}_1 / \mathcal{F}, \dots, \hat{a}_n / \mathcal{F} \in \prod_{i \in I} A_i / \mathcal{F}$, $f^{\prod_{i \in I} \mathcal{A}_i / \mathcal{F}}(\hat{a}_1 / \mathcal{F}, \dots, \hat{a}_n / \mathcal{F}) = f^{\prod_{i \in I} \mathcal{A}_i}(\hat{a}_1, \dots, \hat{a}_n) / \mathcal{F}$.

For the sake of simpler notation, the reduced product will be also denoted as $\prod_{\mathcal{F}}^I \mathcal{A}_i$.

Let \mathcal{F} be a proper filter on I . \mathcal{F} is an *ultrafilter* if, and only if, satisfies any of the following equivalent conditions:

- For every $X \subseteq I$, $X \in \mathcal{F}$ if, and only if, $I \setminus X \notin \mathcal{F}$.
- For every $X, Y \subseteq I$, $X \cup Y \in \mathcal{F}$ if, and only if, $X \in \mathcal{F}$ or $Y \in \mathcal{F}$.

- \mathcal{F} is maximal in the set of proper filters on I ordered by the inclusion.

The reduced product w.r.t. an ultrafilter is called *ultraproduct*.

Given a family $\{\mathcal{A}_i : i \in I\} \cup \{\mathcal{A}\}$ of algebras of the same type, we say that \mathcal{A} is a *subdirect product* of $\{\mathcal{A}_i : i \in I\}$ if, and only if:

1. $\mathcal{A} \subseteq \prod_{i \in I} \mathcal{A}_i$, and
2. for every $j \in I$, the restriction on \mathcal{A} of the j -th projection of $\prod_{i \in I} \mathcal{A}_i$ is surjective.

\mathcal{A} is *representable as a subdirect product* of $\{\mathcal{A}_i : i \in I\}$ if, and only if it is isomorphic to a subdirect product of $\{\mathcal{A}_i : i \in I\}$, i.e. there exists an embedding $\alpha : \mathcal{A} \hookrightarrow \prod_{i \in I} \mathcal{A}_i$ such that for every $j \in J$, $\pi_j \circ \alpha$ is surjective. In this case α is called a *subdirect representation* of \mathcal{A} . We say that the subdirect representation is finite when I is finite.

An algebra \mathcal{A} is *(finitely) subdirectly irreducible* if, and only if, for every (finite) representation $\alpha : \mathcal{A} \hookrightarrow \prod_{i \in I} \mathcal{A}_i$ there exists $j \in J$ such that $\pi_j \circ \alpha$ is an isomorphism.

Proposition 2.1. *Let \mathcal{A} be an algebra and take $\theta \in \text{Con}(\mathcal{A}) \setminus \{\nabla_{\mathcal{A}}\}$. The following are equivalent:*

- (i) \mathcal{A}/θ is subdirectly irreducible.
- (ii) θ is \cap -completely irreducible.
- (iii) θ is maximal relatively to a pair, i.e. there is a pair $\langle a, b \rangle \in A^2$ such that θ is maximal in the set of proper congruences not containing $\langle a, b \rangle$.

Corollary 2.2. *Let \mathcal{A} be an algebra. The following are equivalent:*

- (i) \mathcal{A} is subdirectly irreducible.
- (ii) $\text{Con}(\mathcal{A}) \setminus \{\Delta_{\mathcal{A}}\}$ has a minimum element.

Theorem 2.3 ([15]). *Every algebra \mathcal{A} is representable as a subdirect product of subdirectly irreducible algebras (which are homomorphic images of \mathcal{A}).*

Given a class of algebras \mathbb{K} , we denote the class of its subdirectly irreducible members by \mathbb{K}_{SI} and the class of its finitely subdirectly irreducible members by \mathbb{K}_{FSI} .

Notice that simple algebras are subdirectly irreducible. An algebra is called *semisimple* if, and only if, it is representable as a subdirect product of simple algebras.

The operators over classes of algebras that give their isomorphic images, subalgebras, homomorphic images, products, reduced products and ultraproducts are respectively denoted as **I**, **S**, **H**, **P**, **P_R**, **P_U**.

Given a class of algebras \mathbb{K} of the same type and an operator **O** $\in \{\mathbf{I}, \mathbf{S}, \mathbf{H}, \mathbf{P}, \mathbf{P}_R, \mathbf{P}_U\}$, the following hold:

1. $\mathbf{O}(\mathbb{K}) \subseteq \mathbf{IO}(\mathbb{K})$,
2. $\mathbf{IO}(\mathbb{K}) = \mathbf{OI}(\mathbb{K})$,
3. $\mathbf{IPS}(\mathbb{K}) \subseteq \mathbf{ISP}(\mathbb{K})$,
4. $\mathbf{IPH}(\mathbb{K}) \subseteq \mathbf{HP}(\mathbb{K})$,
5. $\mathbf{ISH}(\mathbb{K}) \subseteq \mathbf{IHS}(\mathbb{K})$, and
6. $\mathbf{ISP}_R(\mathbb{K}) = \mathbf{ISPP}_U(\mathbb{K})$.

If $\mathbb{K} = \{\mathcal{A}_1, \dots, \mathcal{A}_n\}$, we write $\mathbf{O}(\mathcal{A}_1, \dots, \mathcal{A}_n)$ instead of $\mathbf{O}(\{\mathcal{A}_1, \dots, \mathcal{A}_n\})$.

Moreover, \mathbb{K} is said to be a *variety* if, and only if, it is closed under \mathbf{H} , \mathbf{S} and \mathbf{P} . We denote as $\mathbf{V}(\mathbb{K})$ the variety generated by \mathbb{K} , i.e. the smallest variety containing \mathbb{K} . It is clear that $\mathbf{V}(\mathbb{K}) = \mathbf{HSP}(\mathbb{K})$.

Given a formula $\varphi(x_1, \dots, x_n) \in Fm_{\mathcal{L}}$ and an algebra \mathcal{A} of type \mathcal{L} , we define inductively a function $\varphi^{\mathcal{A}} : A^n \rightarrow A$ by:

1. If φ is a variable x_i , $\varphi^{\mathcal{A}}(a_1, \dots, a_n) := a_i$, for every $a_1, \dots, a_n \in A$.
2. If φ is a functional c , with arity 0, $\varphi^{\mathcal{A}}(a_1, \dots, a_n) := c^{\mathcal{A}}$, for every $a_1, \dots, a_n \in A$.
3. If φ is of the form $f(\varphi_1(x_1, \dots, x_n), \dots, \varphi_m(x_1, \dots, x_n))$, $\varphi^{\mathcal{A}}(a_1, \dots, a_n) := f^{\mathcal{A}}(\varphi_1^{\mathcal{A}}(a_1, \dots, a_n), \dots, \varphi_m^{\mathcal{A}}(a_1, \dots, a_n))$, for every $a_1, \dots, a_n \in A$.

An \mathcal{L} -equation is an expression of the form:

$$\varphi \approx \psi$$

where $\varphi, \psi \in Fm_{\mathcal{L}}$. The set of all equations is denoted as $Eq_{\mathcal{L}}$. Given an \mathcal{L} -equation $\varphi(x_1, \dots, x_n) \approx \psi(x_1, \dots, x_n)$ and an algebra \mathcal{A} of type \mathcal{L} , we say that \mathcal{A} satisfies (or verifies) the equation if, and only if, for every $a_1, \dots, a_n \in A$, $\varphi^{\mathcal{A}}(a_1, \dots, a_n) = \psi^{\mathcal{A}}(a_1, \dots, a_n)$. It is denoted as $\mathcal{A} \models \varphi \approx \psi$. A class of algebras \mathbb{K} of type \mathcal{L} satisfies an equation $\varphi \approx \psi \in Eq_{\mathcal{L}}$ if, and only if, $\mathcal{A} \models \varphi \approx \psi$ for every $\mathcal{A} \in \mathbb{K}$. It is denoted as $\mathbb{K} \models \varphi \approx \psi$. \mathbb{K} satisfies a set of equations $\Lambda \subseteq Eq_{\mathcal{L}}$ if, and only if, $\mathbb{K} \models \varphi \approx \psi$ for every $\varphi \approx \psi \in \Lambda$. It is denoted as $\mathbb{K} \models \Lambda$.

Theorem 2.4 ([14]). *Let \mathcal{L} be an algebraic language and let \mathbb{K} be a class of algebras of type \mathcal{L} . \mathbb{K} is a variety if, and only if, it is an equational class (i.e. there exists a set of equations $\Lambda \subseteq Eq_{\mathcal{L}}$ such that $\mathbb{K} = \{\mathcal{A} : \mathcal{A} \models \Lambda\}$).*

We say that an algebra \mathcal{A} is *congruent permutable* if, and only if, for every $\theta_1, \theta_2 \in \text{Con}(\mathcal{A})$, $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. A variety \mathbb{K} is *congruent permutable* if, and only if, for every $\mathcal{A} \in \mathbb{K}$, \mathcal{A} is congruent permutable. A variety \mathbb{K} is *congruent distributive* if, and only if, for every $\mathcal{A} \in \mathbb{K}$, $\mathbf{Con}(\mathcal{A})$ is a distributive lattice. A variety is *arithmetic* if, and only if, it is congruent distributive and congruent permutable.

The finitely subdirectly irreducible members of a congruent distributive variety have a useful description, as the following result states.

Theorem 2.5 (Jónsson's Lemma). *Let \mathbb{K} be a class of algebras of the same type such that $\mathbf{V}(\mathbb{K})$ is congruent distributive. If an algebra $\mathcal{A} \in \mathbf{V}(\mathbb{K})$ is finitely subdirectly irreducible, then $\mathcal{A} \in \mathbf{HSP}_U(\mathbb{K})$.*

A class of algebras \mathbb{K} is said to be a *quasivariety* if, and only if, it is closed under **I**, **S** and **P_R**. We denote as $\mathbf{Q}(\mathbb{K})$ the quasivariety generated by \mathbb{K} , i.e. the smallest quasivariety containing \mathbb{K} . It is clear that $\mathbf{Q}(\mathbb{K}) = \mathbf{ISP}_R(\mathbb{K})$.

An \mathcal{L} -quasiequation is an expression of the form:

$$\varphi_0 \approx \psi_0 \& \dots \& \varphi_{n-1} \approx \psi_{n-1} \Rightarrow \varphi_n \approx \psi_n$$

where $\varphi_i, \psi_i \in Fm_{\mathcal{L}}$ for every $i \leq n$. The set of all quasiequations is denoted as $QE_{\mathcal{L}}$. Notice that $Eq_{\mathcal{L}} \subseteq QE_{\mathcal{L}}$, since an equation is a quasiequation with $n = 0$. Given an \mathcal{L} -quasiequation $\varphi_0 \approx \psi_0 \& \dots \& \varphi_{n-1} \approx \psi_{n-1} \Rightarrow \varphi_n \approx \psi_n$ such that its variables are in $\{x_1, \dots, x_m\}$ and an algebra \mathcal{A} of type \mathcal{L} , we say that \mathcal{A} satisfies the quasiequation if, and only if, for every $a_1, \dots, a_m \in A$, $\varphi_n^{\mathcal{A}}(a_1, \dots, a_m) = \psi_n^{\mathcal{A}}(a_1, \dots, a_m)$, whenever $\varphi_i^{\mathcal{A}}(a_1, \dots, a_m) = \psi_i^{\mathcal{A}}(a_1, \dots, a_m)$ for every $i < n$. It is denoted as $\mathcal{A} \models \varphi_0 \approx \psi_0 \& \dots \& \varphi_{n-1} \approx \psi_{n-1} \Rightarrow \varphi_n \approx \psi_n$. The extension of this definition to classes of algebras and sets of quasiequations is done in the obvious way, as in the case of equations.

Theorem 2.6 ([111]). *Let \mathcal{L} be an algebraic language and let \mathbb{K} be a class of algebras of type \mathcal{L} . \mathbb{K} is a quasivariety if, and only if, it is a quasiequational class (i.e. there exists a set of quasiequations $\Lambda \subseteq QE_{\mathcal{L}}$ such that $\mathbb{K} = \{\mathcal{A} : \mathcal{A} \models \Lambda\}$).*

Since $Eq_{\mathcal{L}} \subseteq QE_{\mathcal{L}}$, the last theorem implies that every variety is a quasivariety.

Let now $Fm_{\mathcal{L}}$ be a set formulae built over a set of variables with arbitrary length. A generalized \mathcal{L} -quasiequation is an expression of the form:

$$\&_{i < \kappa} \varphi_i \approx \psi_i \Rightarrow \varphi_{\kappa} \approx \psi_{\kappa}$$

where $\varphi_i, \psi_i \in Fm_{\mathcal{L}}$ for every $i \leq \kappa$, where κ is a cardinal number. The set of all generalized equations is denoted as $GQE_{\mathcal{L}}$. Of course, $QE_{\mathcal{L}} \subseteq GQE_{\mathcal{L}}$. The satisfaction relations of generalized quasiequations are the obvious generalization of the corresponding notions for quasiequations.

Generalized quasiequations determine a kind of generalized quasivarieties, namely classes of algebras closed under isomorphic images, subalgebras and products, as the following theorem states (cf. [78], page 380).

Theorem 2.7. *Let \mathcal{L} be an algebraic language and let \mathbb{K} be a class of algebras of type \mathcal{L} . \mathbb{K} is closed under **I**, **S** and **P** if, and only if, there exists a set of generalized quasiequations $\Lambda \subseteq GQE_{\mathcal{L}}$ such that $\mathbb{K} = \{\mathcal{A} : \mathcal{A} \models \Lambda\}$.*

A variety \mathbb{K} has the *equationally definable principal congruences property* (EDPC, for short) if, and only if, there exists a finite set of equations in 4 variables

$$\{\sigma_i(x, y, z, w) \approx \tau_i(x, y, z, w) : i < n\}$$

such that for every $\mathcal{A} \in \mathbb{K}$ and every $a, b, c, d \in A$: $\langle c, d \rangle \in \Theta(\{\langle a, b \rangle\})$ if, and only if, $\sigma_i^{\mathcal{A}}(a, b, c, d) = \tau_i^{\mathcal{A}}(a, b, c, d)$ for every $i < n$.

A variety \mathbb{K} has the *congruence extension property* (CEP, for short) if, only if, for every $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ such that $\mathcal{B} \subseteq \mathcal{A}$ and every $\theta \in \text{Con}(\mathcal{B})$, there exists $\theta' \in \text{Con}(\mathcal{A})$ such that $\theta' \cap B^2 = \theta$.

A class \mathbb{K} of algebras is *locally finite* (LF, for short) if, and only if, for every $\mathcal{A} \in \mathbb{K}$ and for every finite set $B \subseteq A$, the subalgebra generated by B is also finite. Notice that this property is inherited by the subclasses of \mathbb{K} .

Let \mathcal{L} be an algebraic language, let $\mathcal{A} = \langle A, \{f^{\mathcal{A}} : f \in F\} \rangle$ be an algebra of type \mathcal{L} and let $B \subseteq A$ be a non-empty set. The *partial subalgebra* \mathcal{B} of \mathcal{A} with domain B is the partial algebra $\langle B, \{f^{\mathcal{B}} : f \in F\} \rangle$, where for every $f \in F$ n -ary, and every $b_1, \dots, b_n \in B$,

$$f^{\mathcal{B}}(b_1, \dots, b_n) = \begin{cases} f^{\mathcal{A}}(b_1, \dots, b_n) & \text{if } f^{\mathcal{A}}(b_1, \dots, b_n) \in B, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We denote it by $\mathcal{B} \subseteq_p \mathcal{A}$.

Given two algebras \mathcal{A} and \mathcal{B} of the same language we say that \mathcal{A} is *partially embeddable* into \mathcal{B} when every finite partial subalgebra of \mathcal{A} is embeddable into \mathcal{B} . Generalizing this notion to classes of algebras, we say that a class \mathbb{K} of algebras is *partially embeddable into a class* \mathbb{M} if every finite partial subalgebra of a member of \mathbb{K} is embeddable into a member of \mathbb{M} .

If the language is finite, this turns out to be equivalent to say that \mathbb{K} belongs to the universal class generated by \mathbb{M} (see for instance [76]). That is, by recalling Los' theorem (see [24]) of characterization of universal classes, we have the following equivalence.

Proposition 2.8 ([76], Th. 1.2.2). *Let \mathbb{K} and \mathbb{M} be classes of algebras of the same finite language. Then the following conditions are equivalent:*

- \mathbb{K} is partially embeddable into \mathbb{M}
- $\mathbb{K} \subseteq \text{ISP}_U(\mathbb{M})$

Given a class \mathbb{K} of algebras, \mathbb{K}_{fin} will denote the class of its finite members.

A class \mathbb{K} of algebras has the *finite embeddability property* (FEP, for short) if, and only if, it is partially embeddable into \mathbb{K}_{fin} .

A class \mathbb{K} of algebras of the same type has the *strong finite model property* (SFMP, for short) if, and only if, every quasiequation that fails to hold in \mathbb{K} can be refuted in some member of \mathbb{K}_{fin} .

A class \mathbb{K} of algebras of the same type has the *finite model property* (FMP, for short) if, and only if, every equation that fails to hold in \mathbb{K} can be refuted in some member of \mathbb{K}_{fin} .

It is clear that a variety has the FMP if, and only if, it is generated by its finite members and a quasivariety has the SFMP if, and only if, it is generated (as a quasivariety) by its finite members.

Theorem 2.9 ([22], Th. 3.1). *Let \mathcal{L} be a finite algebraic language and let \mathbb{K} be a class of algebras of type \mathcal{L} closed under finite products. Then, \mathbb{K} has the FEP if, and only if, \mathbb{K} has the SFMP.*

Moreover, it is clear that for every class of algebras \mathbb{K} , we have:

- If \mathbb{K} is locally finite, then it has the FEP.
- If \mathbb{K} has the FEP, then it has the SFMP.
- If \mathbb{K} has the SFMP, then it has the FMP.

Theorem 2.10 ([22], Th. 3.3, cf. [18]). *Let \mathcal{L} be a finite algebraic language and let \mathbb{K} be a variety of algebras of type \mathcal{L} enjoying the EDPC. Then, the following are equivalent:*

- \mathbb{K} has the FEP,
- \mathbb{K} has the SFMP,
- \mathbb{K} has the FMP.

2.2 Algebraic Logic

Again, regarding to Algebraic Logic, we introduce only the definitions, basic general facts, and notations that are needed in the dissertation. Nevertheless, this is an already far developed subject. The interested reader can find a useful introductory survey to its abstract theory in [62], and reference textbooks in [41] and [61].

A *propositional language* \mathcal{L} is an algebraic language. The functional symbols of \mathcal{L} are called *propositional connectives*. A *propositional logic* (also called *sentential logic*) is a pair $S = \langle \mathcal{L}, \vdash_S \rangle$ where \mathcal{L} is a propositional language and $\vdash_S \subseteq \mathcal{P}(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}}$ satisfying the following conditions:

1. Consequence relation:
 - For every $\Gamma \cup \Delta \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$,
 - (a) If $\varphi \in \Gamma$, then $\Gamma \vdash_S \varphi$.
 - (b) If $\Gamma \vdash_S \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_S \varphi$.
 - (c) If $\Gamma \vdash_S \varphi$ and for every $\psi \in \Gamma$, $\Delta \vdash_S \psi$, then $\Delta \vdash_S \varphi$.
2. Structural:
 - For every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ and every homomorphism $\sigma : \mathbf{Fm}_{\mathcal{L}} \rightarrow \mathbf{Fm}_{\mathcal{L}}$ (these homomorphisms are called *substitutions*), if $\Gamma \vdash_S \varphi$, then $\sigma[\Gamma] \vdash_S \sigma(\varphi)$.

Given $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, we write $\Gamma \vdash_S \varphi$ instead of $\langle \Gamma, \varphi \rangle \in \vdash_S$, and we write $\Gamma \not\vdash_S \varphi$ instead of $\langle \Gamma, \varphi \rangle \notin \vdash_S$.

A formula φ is a *theorem* of S if, and only if, $\emptyset \vdash_S \varphi$. In such a case we write $\vdash_S \varphi$.

A propositional logic $S = \langle \mathcal{L}, \vdash_S \rangle$ is *finitary* if, and only if, $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ such that $\Gamma \vdash_S \varphi$, there exists a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_S \varphi$.

A propositional logic $S = \langle \mathcal{L}, \vdash_S \rangle$ is *decidable* if, and only if, there is an effective process that for every $\varphi \in Fm_{\mathcal{L}}$ decides whether $\vdash_S \varphi$ or $\not\vdash_S \varphi$.

Let \mathcal{L} be a propositional language and \mathbb{K} a class of algebras of type \mathcal{L} . The *equational consequence* associated to \mathbb{K} is defined in the following way: for every $\Lambda \cup \{\varphi \approx \psi\} \subseteq Eq_{\mathcal{L}}$, $\Lambda \models_{\mathbb{K}} \varphi \approx \psi$ iff for every $\mathcal{A} \in \mathbb{K}$ and every homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \rightarrow \mathcal{A}$, it holds:

$$\text{if } h(\alpha) = h(\beta) \text{ for every } \alpha \approx \beta \in \Lambda, \text{ then } h(\varphi) = h(\psi).$$

Then, the following facts can be proved:

- The pair $\langle \mathcal{L}, \models_{\mathbb{K}} \rangle$ satisfies the conditions in the definition of propositional logic, where $Eq_{\mathcal{L}}$ plays the role of $Fm_{\mathcal{L}}$.
- $\models_{\mathbb{K}} = \models_{\mathbf{ISP}(\mathbb{K})}$.
- If $\mathbf{P}_U(\mathbb{K}) \subseteq \mathbb{K}$, then $\langle \mathcal{L}, \models_{\mathbb{K}} \rangle$ is finitary.

A *Hilbert-style calculus* is a triple $H = \langle \mathcal{L}, AX, IR \rangle$, where \mathcal{L} is a propositional language, and the sets $AX \subseteq Fm_{\mathcal{L}}$ (the set of *axioms*) and $IR \subseteq \bigcup_{n \leq 1} (Fm_{\mathcal{L}})^n$ (the set of *inference rules*) are closed under substitutions.¹ H defines a relation $\vdash_H \subseteq \mathcal{P}(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}}$ as follows:

Given $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_H \varphi$ iff there exists a finite sequence of formulae $\langle \varphi_0, \dots, \varphi_n \rangle$ such that:

- $\varphi_n = \varphi$, and
- for every $i \leq n$, either $\varphi_i \in \Gamma \cup AX$ or there is $\langle \psi_0, \dots, \psi_m \rangle \in IR$ such that $\psi_m = \varphi_i$ and $\{\psi_0, \dots, \psi_{m-1}\} \subseteq \{\varphi_0, \dots, \varphi_{i-1}\}$.

In such a case we say that φ is *derivable* from Γ in the calculus H .

The pair $\langle \mathcal{L}, \vdash_H \rangle$ is a finitary propositional logic.

Theorem 2.11 ([109]). *Let $S = \langle \mathcal{L}, \vdash_S \rangle$ be a propositional logic. Then, S is finitary if, and only if, there exists a Hilbert-style calculus H such that $\vdash_H = \vdash_S$.*

A finitary logic is *finitely axiomatizable* if, and only if, it is equivalent to a Hilbert-style calculus which can be presented by using a finite number of schemata.

¹Usually the sets AX and IR are presented by using schemata, i.e. by showing a particular formula (or rule) and assuming that all its substitutions are also included in the set. For instance, to say that the system has the Modus Ponens, we write that the schema $\langle p, p \rightarrow q, q \rangle$ is in IR , but this will actually mean that all the substitutions of this schema are also in IR .

Given two propositional languages \mathcal{L} and \mathcal{L}' such that $\mathcal{L} \subseteq \mathcal{L}'$ and two propositional logics $S = \langle \mathcal{L}, \vdash_S \rangle$ and $S' = \langle \mathcal{L}', \vdash_{S'} \rangle$, we say that S' is an *expansion* of S if, and only if, $\vdash_S \subseteq \vdash_{S'}$. The expansion is *conservative* if, and only if, for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_S \varphi$ iff $\Gamma \vdash_{S'} \varphi$; in this case we say that S is the \mathcal{L} -fragment of S' . We say that S' is an *extension* of S if, and only if, $\vdash_S \subseteq \vdash_{S'}$ and $\mathcal{L} = \mathcal{L}'$.

Let $H = \langle \mathcal{L}, AX, IR \rangle$ and $H' = \langle \mathcal{L}', AX', IR' \rangle$ be two Hilbert-style calculi, and let S and S' be their corresponding sentential logics. We say that S' is an axiomatic extension (resp. expansion) of S if, and only if, $AX \subseteq AX'$, $IR = IR'$ and $\mathcal{L} = \mathcal{L}'$ (resp. $\mathcal{L} \subseteq \mathcal{L}'$).

Let $S = \langle \mathcal{L}, \vdash_S \rangle$ be a finitary propositional logic and \mathbb{K} a class of algebras of type \mathcal{L} . \mathbb{K} is an *algebraic semantics* for S if, and only if, there is a finite set of \mathcal{L} -equations in one variable

$$\delta(x) \approx \varepsilon(x) = \{\delta_i(x) \approx \varepsilon_i(x) : i < n\},$$

which is called *system of defining equations*, such that for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_S \varphi$ iff $\{\delta(\psi) \approx \varepsilon(\psi) : \psi \in \Gamma\} \models_{\mathbb{K}} \delta(\varphi) \approx \varepsilon(\varphi)$

Proposition 2.12 ([19], Cor. 2.3). *If \mathbb{K} is an algebraic semantics for a finitary propositional logic S , then $\mathbf{Q}(\mathbb{K})$ is also an algebraic semantics for S and it has the same system of defining equations.*

Definition 2.13 ([19]). *Let $S = \langle \mathcal{L}, \vdash_S \rangle$ be a finitary propositional logic and \mathbb{K} a class of algebras of type \mathcal{L} . \mathbb{K} is an equivalent algebraic semantics for S if, and only if, it is an algebraic semantics for S and there is a finite set of formulae in two variables $\Delta(x, y) = \{\Delta_j(x, y) : j < m\}$, which are called equivalence formulae, such that for every $\varphi \approx \psi \in Eq_{\mathcal{L}}$ it holds:*

- $\varphi \approx \psi \models_{\mathbb{K}} \delta(\Delta(\varphi, \psi)) \approx \varepsilon(\Delta(\varphi, \psi))$, and
- $\delta(\Delta(\varphi, \psi)) \approx \varepsilon(\Delta(\varphi, \psi)) \models_{\mathbb{K}} \varphi \approx \psi$,

where $\delta \approx \varepsilon$ is the system of defining equations of \mathbb{K} and $\delta(\Delta(\varphi, \psi)) \approx \varepsilon(\Delta(\varphi, \psi))$ is a shorthand for $\{\delta_i(\Delta_j(\varphi, \psi)) \approx \varepsilon_i(\Delta_j(\varphi, \psi)) : i < n, j < m\}$.

A finitary propositional logic is algebraizable if, and only if, it has an equivalent algebraic semantics.

Proposition 2.14 ([19], Cor. 2.11). *Let \mathbb{K} be an equivalent algebraic semantics for a finitary propositional logic S . Then, \mathbb{K} is an equivalent algebraic semantics for S iff $\mathbf{Q}(\mathbb{K})$ is also an equivalent algebraic semantics for S .*

Theorem 2.15 ([19], Th. 2.15). *Let $S = \langle \mathcal{L}, \vdash_S \rangle$ be a finitary propositional logic. If \mathbb{K} and \mathbb{K}' are two equivalent algebraic semantics for S , then $\mathbf{Q}(\mathbb{K}) = \mathbf{Q}(\mathbb{K}')$.*

This quasivariety is called the *equivalent quasivariety semantics* of S . It can be axiomatized as the following theorem describes.

Theorem 2.16 ([19], Th. 2.17). *Let $S = \langle \mathcal{L}, \vdash_S \rangle$ be an algebraizable logic and let $H = \langle \mathcal{L}, AX, IR \rangle$ be an equivalent Hilbert-style calculus. Let $\delta \approx \varepsilon$ be the system of defining equations and $\Delta(x, y)$ the equivalence formulae. Then, the equivalent quasivariety semantics is axiomatizable by the following quasiequations:*

1. $\delta(\varphi) \approx \varepsilon(\varphi)$, for each $\varphi \in AX$,
2. $\delta(\Delta(x, x)) \approx \varepsilon(\Delta(x, x))$,
3. $\delta(\varphi_0) \approx \varepsilon(\varphi_0) \& \dots \& \delta(\varphi_{n-1}) \approx \varepsilon(\varphi_{n-1}) \Rightarrow \delta(\varphi_n) \approx \varepsilon(\varphi_n)$, for each $\langle \varphi_0, \dots, \varphi_n \rangle \in IR$, and
4. $\delta(\Delta(x, y)) \approx \varepsilon(\Delta(x, y)) \Rightarrow x \approx y$.

Theorem 2.17 (cf. [19]). *Let $S = \langle \mathcal{L}, \vdash_S \rangle$ be an algebraizable logic and let \mathbb{K} be its equivalent quasivariety semantics. Let $\delta \approx \varepsilon$ be the system of defining equations and $\Delta(x, y)$ the equivalence formulae. Then, every finitary extension of S is algebraizable (with the same defining equations and equivalence formulae) and we have the following dual order isomorphism between the lattice of subquasivarieties of \mathbb{K} and the lattice of finitary extensions of S :*

1. *If $\Gamma \subseteq Fm_{\mathcal{L}}$, $\Sigma \subseteq \bigcup_{n \leq 1} (Fm_{\mathcal{L}})^n$ are closed under substitutions and S' is the extension of S obtained by adding the formulae of Γ as axioms and adding Σ as inference rules, then the equivalent algebraic semantics of S' is the subquasivariety of \mathbb{K} axiomatized by the quasiequations*
 - $\delta(\varphi) \approx \varepsilon(\varphi)$, for each $\varphi \in \Gamma$, and
 - $\delta(\varphi_0) \approx \varepsilon(\varphi_0) \& \dots \& \delta(\varphi_{n-1}) \approx \varepsilon(\varphi_{n-1}) \Rightarrow \delta(\varphi_n) \approx \varepsilon(\varphi_n)$, for each $\langle \varphi_0, \dots, \varphi_n \rangle \in \Sigma$.
2. *Let $\mathbb{K}' \subseteq \mathbb{K}$ be the subquasivariety axiomatized by a set of quasiequations Λ . Then the logic associated to \mathbb{K}' is the finitary extension of S given by the rules $\{ \langle \Delta(\varphi_0, \psi_0), \dots, \Delta(\varphi_n, \psi_n) \rangle : \varphi_0 \approx \psi_0 \& \dots \& \varphi_{n-1} \approx \psi_{n-1} \Rightarrow \varphi_n \approx \psi_n \in \Lambda \}$.*

When the equivalent algebraic semantics is a variety, by restricting the dual order isomorphism of the last theorem, we also obtain a bijective correspondence between axiomatic extensions and subvarieties.

On the one hand, it is clear that every fragment of an algebraizable logic whose language contains all the connectives occurring in the defining equations and in the equivalence formulae, is also algebraizable. On the other hand, some expansions of an algebraizable logic are algebraizable as well. Indeed, let $S = \langle \mathcal{L}, \vdash_S \rangle$ be an algebraizable logic and let \mathbb{K} be its equivalent quasivariety semantics. Let $\delta \approx \varepsilon$ be the system of defining equations and $\Delta(x, y)$ the equivalence formulae. Let \mathcal{L}' be a language extending \mathcal{L} and $S' = \langle \mathcal{L}', \vdash_{S'} \rangle$ be the expansion of S obtained by adding some sets Γ as axioms and Σ as inference rules. Assume that for every new n -ary connective λ in the language \mathcal{L}' ,

$$\Delta(x_1, y_1) \cup \dots \cup \Delta(x_n, y_n) \vdash_{\mathcal{L}'} \Delta(\lambda(x_1, \dots, x_n), \lambda(y_1, \dots, y_n))$$

Then, S' is algebraizable and its equivalent quasivariety semantics \mathbb{K}' is axiomatized by the axioms of \mathbb{K} plus the quasiequations

- $\delta(\varphi) \approx \varepsilon(\varphi)$, for each $\varphi \in \Gamma$, and
- $\delta(\varphi_0) \approx \varepsilon(\varphi_0) \& \dots \& \delta(\varphi_{n-1}) \approx \varepsilon(\varphi_{n-1}) \Rightarrow \delta(\varphi_n) \approx \varepsilon(\varphi_n)$, for each $\langle \varphi_0, \dots, \varphi_n \rangle \in \Sigma$.

In general, S' needs not be a conservative expansion of S .

Proposition 2.18 (cf. [19]). *Under the previous hypothesis, S' is a conservative expansion of S if, and only if, every algebra of \mathbb{K} is a subreduct of \mathbb{K}' .*

There is an intrinsic characterization of algebraizable logics, i.e. with no reference to its semantics.

Theorem 2.19 ([19], Th. 4.7). *A sentential logic $S = \langle \mathcal{L}, \vdash_S \rangle$ is algebraizable if, and only if, there is a set of formulae in two variables $\Delta(x, y) \subseteq Fm_{\mathcal{L}}$ and a system of equations in one variable $\delta \approx \varepsilon \subseteq Eq_{\mathcal{L}}$ such that the following conditions hold:*

- (i) $\vdash_S \Delta(x, x)$
- (ii) $\Delta(x, y) \vdash_S \Delta(y, x)$
- (iii) $\Delta(x, y) \cup \Delta(y, z) \vdash_S \Delta(x, z)$
- (iv) $x \vdash_S \Delta(\delta(x), \varepsilon(x))$ and $\Delta(\delta(x), \varepsilon(x)) \vdash_S x$

For every n -ary λ in the language \mathcal{L} ,

- (v) $\Delta(x_1, y_1) \cup \dots \cup \Delta(x_n, y_n) \vdash_S \Delta(\lambda(x_1, \dots, x_n), \lambda(y_1, \dots, y_n))$

In this case $\delta \approx \varepsilon$ is the system of defining equations and $\Delta(x, y)$ are the equivalence formulae.

Using the equivalence formulae it is also possible to translate the axiomatization of the equivalent quasivariety semantics into a Hilbert-style calculus for the algebraizable logic. Therefore, we obtain the next result.

Theorem 2.20. *Let $S = \langle \mathcal{L}, \vdash_S \rangle$ be an algebraizable logic and let \mathbb{K} be its equivalent quasivariety semantics. Then, S is finitely axiomatizable if, and only if, \mathbb{K} is finitely axiomatizable.*

This kind of results, like the last one, connecting properties of the algebraizable logic with properties of its corresponding quasivariety are called *bridge theorems* and show the power of Algebraic Logic. We will state the ones that will be needed in the dissertation. For instance, notice that if \mathbb{K} is the equivalent quasivariety semantics of a logic S and \mathbb{K} enjoys the FMP, then S is decidable.

Other logical properties which have their equivalent algebraic property are the Deduction-Detachment Theorem and the Local Deduction-Detachment Theorem.

A finitary propositional logic $S = \langle \mathcal{L}, \vdash_S \rangle$ has the *Deduction-Detachment Theorem* (DDT, for short) if, and only if, there is a finite set of formulae in two variables $E(x, y)$ such that for every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \cup \{\varphi\} \vdash_S \psi$ iff $\Gamma \vdash_S E(\varphi, \psi)$.

Theorem 2.21 ([21], Th. 5.4). *Let $S = \langle \mathcal{L}, \vdash_S \rangle$ be an algebraizable logic and let \mathbb{K} be its equivalent algebraic semantics. Suppose that \mathbb{K} is a variety. Then, S has the DDT if, only if, \mathbb{K} has the EDPC.*

A finitary propositional logic $S = \langle \mathcal{L}, \vdash_S \rangle$ has the *Local Deduction-Detachment Theorem* (LDDT, for short) if, and only if, there is a family of finite sets of formulae in two variables $\{E_i(x, y) : i \in I\}$ such that for every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \cup \{\varphi\} \vdash_S \psi$ iff there is $i \in I$ such that $\Gamma \vdash_S E_i(\varphi, \psi)$.

Theorem 2.22 (cf. [20]). *Let $S = \langle \mathcal{L}, \vdash_S \rangle$ be an algebraizable logic and let \mathbb{K} be its equivalent algebraic semantics. Suppose that \mathbb{K} is a variety. Then, S has the LDDT if, only if, \mathbb{K} has the CEP.*

2.3 Some results on ordered Abelian groups

In this section we list some results on ordered Abelian groups that are used in the dissertation. Some of them are particular cases of most general results about ordered Abelian groups but we give the results we need and some of the proofs for the reader's convenience.

For the first result we recall the definition of the ordered group obtained as the lexicographic product of copies of the o.a.g of the positive real numbers with the natural order and the product operation $(\mathbb{R}^+, \cdot, \leq)$. For any natural k , we denote by $(\mathcal{R}^+)_{lex}^k = ((\mathbb{R}^+)^k, \bullet, (1, \dots, 1), \leq_{lex})$ the linearly ordered Abelian group defined on the Cartesian product of k copies of the positive reals \mathbb{R}^+ , with \bullet being the coordinatewise multiplication and with \leq_{lex} the lexicographic order. Note that the $\Rightarrow \bullet$ operation in the Π -algebra $\mathbf{P}(\mathcal{R}^+_{lex}^k)$, with $(0, \dots, 0)$ as bottom element, is defined as follows:

$$(a_1, \dots, a_k) \Rightarrow \bullet (b_1, \dots, b_k) = \begin{cases} (1, \dots, 1), & \text{if } (a_1, \dots, a_k) \leq_{lex} (b_1, \dots, b_k) \\ (1, \dots, 1, b_j/a_j, \dots, b_k/a_k), & \text{otherwise} \end{cases}$$

where j is the smallest index for which $a_j > b_j$.

The following result is a consequence of well-known Hahn's theorem, which is a more general result (see e.g. [73, Theorem 4.C]). However, a direct proof of the next theorem can be found in [84, Theorem 7.3.15].

Theorem 2.23. *If G is a finitely generated ordered Abelian group, then G is isomorphic to a subgroup of $(\mathcal{R}^+)_{lex}^k$.*

The second result is given in the following lemma.

Lemma 2.24. *Let H be a subgroup of \mathbb{R}^+ . Any function $t : H \cap (0, 1] \rightarrow \mathbb{R}^+$ such that $t(x \cdot y) = t(x) \cdot t(y)$ for all x, y in $H \cap (0, 1]$ may be extended to a group homomorphism $t' : H \rightarrow \mathbb{R}^+$.*

Proof: Define $t' : H \rightarrow \mathbb{R}^+$ as follows:

$$t'(x) = \begin{cases} t(x), & \text{if } x \leq 1 \\ 1/t(1/x), & \text{if } x > 1 \end{cases}$$

For any $x, y \in H$, one (and only one) of the identities

$$\begin{aligned} t(x \cdot y) &= t(x) \cdot t(y) \\ t(x \cdot y) \cdot t(1/x) &= t(y) \\ t(x \cdot y) \cdot t(1/y) &= t(x) \\ t(1/y) &= t(1/x \cdot 1/y) \cdot t(x) \\ t(1/x) &= t(1/x \cdot 1/y) \cdot t(y) \\ t(1/x) \cdot t(1/y) &= t(1/x \cdot 1/y) \end{aligned}$$

is well defined and satisfied, and implies a corresponding identity with t' instead of t . Since for every $z \in H$ $t'(1/z) \cdot t'(z) = 1$, we may derive $t'(x \cdot y) = t'(x) \cdot t'(y)$ in each of the cases. \square

Finally the third result is a consequence of the fact that an Abelian group is injective² if and only if it is divisible, see e.g. [107, Prop. 3, Sect. 4.2]. Since \mathbb{R}^+ is abelian and divisible, the next lemma follows. For the reader's convenience, we provide a simple elementary proof of the particular case we need.

Lemma 2.25. *Let H be a subgroup of \mathbb{R}^+ , $x \in \mathbb{R}^+ \setminus H$ and let H' be the subgroup generated by H and x . Then every homomorphism $t : H \rightarrow \mathbb{R}^+$ may be extended to a homomorphism $t' : H' \rightarrow \mathbb{R}^+$.*

Proof: If there is no $n \geq 1$ such that $x^n \in H$, then define $t'(x)$ arbitrarily. Every element of H' has a unique decomposition as $x^i \cdot a$, where i is an integer and $a \in H$, so we may define $t'(x^i \cdot a) = t'(x)^i \cdot t(a)$ and this yields a homomorphism on H' .

If there is some $n \geq 1$ such that $x^n \in H$, denote by n the smallest natural number with this property. For every integer i , we have $x^i \in H$ iff n divides i . For every integer i and every $a \in H$ define $t'(x^i \cdot a) = t(x^n)^{i/n} \cdot t(a)$. Let us prove that this is a correct definition. If $x^i \cdot a = x^j \cdot b$ for integers i, j and $a, b \in H$, then there is a natural k such that $j = i - k \cdot n$ and $b = a \cdot x^{kn}$. It follows that $t(x^n)^{j/n} \cdot t(b) = t(x^n)^{i/n-k} \cdot t(a) \cdot t(x^n)^k = t(x^n)^{i/n} \cdot t(a)$. Moreover, the mapping t' is clearly a homomorphism on H' . \square

²An algebra A (or more generally, an object in a category) is injective provided that if B is an algebra of the same class, then every homomorphism from a subalgebra S of B into A can be extended to a homomorphism from the whole B to A .

Part I

MTL and its axiomatic extensions

Chapter 3

The Monoidal Triangular norm based Logic (MTL)

The object of the present study is the logic MTL, whose name is a shorthand for *Monoidal T-norm based Logic*. It was defined by Esteva and Godo in [51] by means of a Hilbert-style calculus in the language $\mathcal{L} = \{\&, \rightarrow, \wedge, \bar{0}\}$ of type $\langle 2, 2, 2, 0 \rangle$. The only inference rule is *Modus Ponens* and the axiom schemata are the following (taking \rightarrow as the least binding connective):

- (A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
- (A2) $\varphi \& \psi \rightarrow \varphi$
- (A3) $\varphi \& \psi \rightarrow \psi \& \varphi$
- (A4) $\varphi \wedge \psi \rightarrow \varphi$
- (A5) $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$
- (A6) $\varphi \& (\varphi \rightarrow \psi) \rightarrow \varphi \wedge \psi$
- (A7a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \& \psi \rightarrow \chi)$
- (A7b) $(\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
- (A8) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
- (A9) $\bar{0} \rightarrow \varphi$

The usual defined connectives are introduced as follows:

$$\varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi);$$

$$\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi);$$

$$\neg \varphi := \varphi \rightarrow \bar{0};$$

$$\bar{1} := \neg \bar{0}.$$

MTL was defined in order to generalize an already defined logic, namely the logic BL introduced by Petr Hájek in [79]. It is also defined by a Hilbert style calculus in the language $\{\&, \rightarrow, \bar{0}\}$ of type $\langle 2, 2, 0 \rangle$. The only inference rule is again *Modus Ponens* and the axiom schemata¹ are the following:

¹In [36] it is proved that the axiom (A3) and the axiom (B3) are redundant in the systems

- (B1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
- (B2) $\varphi \& \psi \rightarrow \varphi$
- (B3) $\varphi \& \psi \rightarrow \psi \& \varphi$
- (B4) $\varphi \& (\varphi \rightarrow \psi) \rightarrow \psi \& (\psi \rightarrow \varphi)$
- (B5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \& \psi \rightarrow \chi)$
- (B5b) $(\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
- (B6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
- (B7) $\bar{0} \rightarrow \varphi$

Now the conjunction \wedge is a defined connective:

$$\varphi \wedge \psi := \varphi \& (\varphi \rightarrow \psi).$$

In [51] it is proved that BL is the extension of MTL obtained by adding the divisibility axiom:

$$\varphi \wedge \psi \rightarrow \varphi \& (\varphi \rightarrow \psi) \quad (\text{Div})$$

Moreover, in [79] Hájek also proved that three well-known many-valued logics can be presented as axiomatic extensions of BL. Indeed, Łukasiewicz logic \mathbf{L} is the extension obtained by adding the involution axiom:

$$\neg \neg \varphi \rightarrow \varphi \quad (\text{Inv})$$

Gödel-Dummett logic \mathbf{G} is the extension obtained by adding the contraction axiom:

$$\varphi \rightarrow \varphi \& \varphi \quad (\text{Con})$$

and Product logic \mathbf{II} is obtained by adding two axiom schemata; one for the cancellation law:

$$\neg \neg \chi \rightarrow ((\varphi \& \chi \rightarrow \psi \& \chi) \rightarrow (\varphi \rightarrow \psi)) \quad (\text{II1}),$$

and one for pseudocomplementation:

$$\varphi \wedge \neg \varphi \rightarrow \bar{0} \quad (\text{II2}) \text{ or } (\text{PC})$$

3.1 MTL as a substructural logic

MTL can be seen as an axiomatic extension of *Monoidal Logic* (ML, for short). This logic was introduced by Höhle in [87].² He gave a Hilbert-style calculus for ML which consisted in 14 axiom schemata and Modus Ponens as the only inference rule in the language \mathcal{L} with an additional binary connective for the disjunction \vee . This axiomatics was simplified by Gottwald, García-Cerdaña and Bou (see [76] and [77]), obtaining the following system:

for MTL and BL, respectively.

²The aim of this logic was to cope with what Höhle believed is the basic structure behind many-valued logics: the lattice ordered residuated monoids (these structures are introduced in the next section of this chapter).

- (Ax_{ML}1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
- (Ax_{ML}2) $\varphi \& \psi \rightarrow \varphi$
- (Ax_{ML}3) $\varphi \& \psi \rightarrow \psi \& \varphi$
- (Ax_{ML}4) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \& \psi \rightarrow \chi)$
- (Ax_{ML}5) $(\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
- (Ax_{ML}6) $\varphi \wedge \psi \rightarrow \varphi$
- (Ax_{ML}7) $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$
- (Ax_{ML}8) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi \wedge \chi))$
- (Ax_{ML}9) $\bar{0} \rightarrow \varphi$
- (Ax_{ML}10) $\varphi \rightarrow \varphi \vee \psi$
- (Ax_{ML}11) $\psi \rightarrow \varphi \vee \psi$
- (Ax_{ML}12) $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi))$

and Modus Ponens as inference rule.

In fact, MTL is the axiomatic extension of ML obtained by adding the schema of prelineality:

$$(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$$

On the other hand, ML turns out to be equivalent to one of the so-called *substructural logics*, namely the logic H_{BCK} of Ono and Komori ([127]), which afterwards Ono has called FL_{ew} (see [126]).

Substructural logics are those where some of the following shemata are not provable:

- (C) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$ (Exchange)
- (K) $\psi \rightarrow (\varphi \rightarrow \psi)$ (Weakening)
- (W) $(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$ (Contraction)

In ML, (K) and (C) are provable, but (W) is not a theorem. The same situation holds in MTL, BL, Π and L . Therefore, these logics can be seen inside the family of substructural logics without contraction. On the contrary, G does prove all these formulae, hence it is not a substructural logic.

In substructural logics the language is richer than that of classical logic. There is terminological distinction between two groups of connectives: additive connectives and multiplicative connectives. In MTL (and its axiomatic extensions) the distincion is the following

1. Multiplicative connectives: $\&, \rightarrow, \leftrightarrow, \neg, \bar{0}, \bar{1}$
2. Additives connectives: $\wedge, \vee, \bar{0}, \bar{1}$.

Sometimes the multiplicative conjunction $\&$ is also called *fusion*.

Since in our approach finitary logics are presented by means of Hilbert-style calculi, here we have not followed the usual definition of substructural logic, but a rather indirect one. Indeed, substructural logics are usually presented by using Gentzen-style calculi, in terms of sequents or hypersequents, where some of the structural Gentzen rules (Exchange, Weakening or Contraction) are not valid. For instance, FL_{ew} was introduced by means of a sequent calculus extending the Full Lambek calculus with the rules of Exchange and Weakening.

The extension of FL_{ew} with Contraction is the Intuitionistic logic. Therefore, in our framework of axiomatic extensions of MTL the only structural logics are those enjoying Contraction, equivalently those that extend the Intuitionistic logic, i.e. Gödel-Dummett logic and its axiomatic extensions.

Some works have been done for MTL and its extensions from the perspective of substructural logics (which is strongly connected to Proof Theory); see for instance [8, 9, 115]. However, in this dissertation we will not consider this aspect of MTL.

3.2 MTL as an algebraizable many-valued logic

As mentioned in the Introduction, the study of many-valued logics began in 1918 and 1922 with the definition of Łukasiewicz's n -valued logics. In 1930 Łukasiewicz and Tarski in [110] introduce also the infinitely-valued version. Other pionnering examples of many-valued logics are introduced by Emil Post (in 1921, [129]), Kleene (in 1938, [101]) and Bochvar (in 1939) with different motivations.

Many-valued logics are usually semantically defined by means of some algebra of truth-values with a set of distinguished elements. Sometimes this semantics is enough to deal with the logic, but other times it is necessary to introduce an alternative semantics. Łukasiewicz logic is an example of the latter; it was formerly introduced in terms of an algebra defined over $[0, 1]$ with 1 as distinguished element, but afterwards some classes of algebras were introduced in order to prove the completeness with respect to Łukasiewicz's calculus: MV-algebras in [25, 26] and their polynomially equivalent form of Wajsberg algebras in [130, 63]. In a similar way, Gödel-Dummett logic was given a semantics in terms of a variety of Heyting algebras. In many cases (including Łukasiewicz and Gödel-Dummett logics) the relation between the many-valued logic and its algebraic semantics is very strong, i.e. they are sometimes algebraizable logics in the sense of Blok and Pigozzi and the semantics is actually their equivalent quasivariety semantics. We will show now that this is the case of MTL, because it is an axiomatic extension of an algebraizable many-valued logic, the Monoidal Logic. To this end we need to introduce the corresponding class of algebras, the residuated lattices.

Definition 3.1. *An integral commutative bounded residuated lattice is an algebra $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ such that:*

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ is a bounded lattice.
2. $\langle A, \&^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ is a commutative monoid.
3. The operations $\&^{\mathcal{A}}$ and $\rightarrow^{\mathcal{A}}$ form an adjoint pair:

$$\forall a, b, c \in A, a \&^{\mathcal{A}} b \leq c \text{ iff } b \leq a \rightarrow^{\mathcal{A}} c.$$

We call it residuated lattice for short. Usually an additional unary operation is defined as $\neg^{\mathcal{A}} a := a \rightarrow^{\mathcal{A}} \bar{0}^{\mathcal{A}}$ for every $a \in A$.

Krull was the first one to study these structures, in [106]. They were called *residuated lattices* for the first time by Dilworth and Ward in [43]. Afterwards they have been studied also under several names: *integral commutative residuated l -monoids* (Birkhoff [16] and Höhle [87]), *BCK-algebras with condition (S)* (Iseki [95]), *BCK-lattices* (Idziak [93, 94]), *full BCK-algebras* (Ono and Komori [127]) and *FL_{ew} -algebras* (Ono [126]).

We denote the class of all residuated lattices by \mathbb{RL} . It is a variety. For instance, the following is an equational base for \mathbb{RL} :

1. $(x \wedge y) \wedge z \approx x \wedge (y \wedge z)$
2. $(x \vee y) \vee z \approx x \vee (y \vee z)$
3. $x \wedge y \approx y \wedge x$
4. $x \vee y \approx y \vee x$
5. $x \wedge x \approx x$
6. $x \vee x \approx x$
7. $x \wedge (x \vee y) \approx x$
8. $x \vee (x \wedge y) \approx x$
9. $x \wedge \bar{0} \approx \bar{0}$
10. $x \vee \bar{1} \approx \bar{1}$
11. $(x \& y) \& z \approx x \& (y \& z)$
12. $x \& y \approx y \& x$
13. $x \& \bar{1} \approx x$
14. $x \& (y \vee z) \approx (x \& y) \vee (x \& z)$
15. $x \& y \rightarrow z \approx x \rightarrow (y \rightarrow z)$
16. $(x \& (x \rightarrow y)) \wedge y \approx x \& (x \rightarrow y)$
17. $x \wedge y \rightarrow y \approx \bar{1}$

Proposition 3.2. *Let $\mathcal{A} \in \mathbb{RL}$. Then:*

- (i) *For every $a, b \in A$, $a \rightarrow b = \bar{1}$ iff $a \leq b$.*
- (ii) *For every $a, b \in A$, $a \rightarrow b = \max\{c \in A : a \& c \leq b\}$.*
- (iii) *$\&$ is a left-continuous operation with respect to the order topology, thus for every $X \subseteq A$ such that there exists $\sup X$ in A , $b \& (\sup X) = \sup\{b \& a : a \in X\}$.*

The algebraic counterpart of MTL logic is a class of algebraic structures called *MTL-algebras*. They are defined as follows.

Definition 3.3 ([51]). *Let $\mathcal{A} \in \mathbb{RL}$. \mathcal{A} is an MTL-algebra iff it satisfies the prelinearity equation:*

$$(x \rightarrow y) \vee (y \rightarrow x) \approx \bar{1}$$

If the lattice order is total we say that \mathcal{A} is an MTL-chain.

MTL will denote the class of all MTL-algebras, i.e. the subvariety of \mathbb{RL} defined by the prelinearity equation.

Adillon and Verdú prove in [1] that H_{BCK} (and hence ML) is an algebraizable logic whose equivalent algebraic semantics is \mathbb{RL} . Therefore, since MTL is the axiomatic extension of ML obtained by adding the prelinearity axiom and after Theorem 2.17, we obtain the following result:³

Theorem 3.4. *MTL is an algebraizable logic and MTL is its equivalent algebraic semantics with the same translations that Adillon and Verdú use for H_{BCK} . Furthermore, all axiomatic extensions of MTL are also algebraizable and their equivalent algebraic semantics are the subvarieties of MTL defined by the translations of the axioms into equations. In particular, there is a dual order isomorphism between axiomatic extensions of MTL and subvarieties of MTL:*

1. *If $\Sigma \subseteq \text{Fm}_{\mathcal{L}}$ and L is the extension of MTL obtained by adding the formulae of Σ as schemata, then the equivalent algebraic semantics of L is the subvariety of MTL axiomatized by the equations $\{\varphi \approx \bar{1} : \varphi \in \Sigma\}$. We denote this variety by \mathbb{L} and we call its members *L-algebras*. There are two main exceptions to that rule: the algebras associated to L are called MV-algebras⁴ following the terminology of Chang in [25], and the algebras associated to the Classical Propositional Calculus (CPC for short) are called, of course, Boolean algebras (\mathbb{BA} will denote the variety of Boolean algebras).*
2. *Let $\mathbb{L} \subseteq \text{MTL}$ be the subvariety axiomatized by a set of equations Λ . Then the logic associated to \mathbb{L} is the axiomatic extension L of MTL given by the axiom schemata $\{\varphi \leftrightarrow \psi : \varphi \approx \psi \in \Lambda\}$.*

It will be useful later on to recall now the definition of some examples of MTL-algebras:

- \mathcal{B}_2 and \mathcal{B}_4 will be the Boolean algebras of two elements and four elements respectively, with the usual definitions.

³Actually, a much more general class of residuated lattices (satisfying neither commutativity nor integrality) is known to be the quasivariety semantics of the logic corresponding to the Full Lambek calculus. Our study of axiomatic extensions of MTL can be seen as a part of the study of algebraizable many-valued logics based on quasivarieties of residuated lattices in this general sense (see e.g. [67], [11] and [68]).

⁴They are also called *Wajsberg algebras* in the polynomially equivalent definition of [130, 63].

- For every $n \geq 3$, \mathbf{L}_n is the MV-algebra defined over the set $\{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$. The operations of strong conjunction and negation in all these algebras have the following expressions: $a \& b = \max\{a + b - 1, 0\}$ and $\neg a = 1 - a$. The remaining operations are defined from the former in the following way: $a \rightarrow b := \neg(a \& \neg b)$, $a \wedge b := a \& (a \rightarrow b)$ and $a \vee b := (a \rightarrow b) \rightarrow b$.
- Chang's algebra \mathcal{C} is another useful example of MV-algebra which is defined by Chang in [25] (page 474). With our notation it can be defined in the following way. Consider the set of rational numbers $C = \{(\frac{1}{2})^n : n \in \omega\} \cup \{-(\frac{1}{2})^n : n \in \omega\}$, endowed with the natural ordering and the following monoidal operation:

$$a \& b := \begin{cases} ab & \text{if } a, b > 0 \\ -\min\{1, -\frac{b}{a}\} & \text{if } a > 0, b < 0 \\ -\min\{1, -\frac{a}{b}\} & \text{if } a < 0, b > 0 \\ -\frac{1}{2} & \text{if } a, b < 0 \end{cases}$$

and its residuum.

- For every $n \geq 3$, \mathcal{G}_n is the G-algebra defined over the set $\{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$. the operations in all these algebras have the following expressions: $a \& b = a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ and

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

3.3 MTL as a t-norm based fuzzy logic

Even though MTL turns out to be a substructural logic and also an algebraizable many-valued logic, originally it was not introduced from these perspectives. Indeed, it was proposed in the framework of triangular norm based fuzzy logics, i.e. as a logic whose intended semantics is a set of algebras defined over the real unit interval $[0, 1]$ where the multiplicative conjunction $\&$ is interpreted by a triangular norm (t-norm, for short) and the implication is its residuum. More formally, a logic $S = \langle \mathcal{L}, \vdash_S \rangle$ is *t-norm based* if, and only if, there exists a set of algebras \mathbb{K} of type \mathcal{L} whose carrier is $[0, 1]$, their interpretation of $\&$ is a t-norm and their interpretation of \rightarrow is its residuum, and such that it holds the following *standard completeness* theorem: for every formula $\varphi \in Fm_{\mathcal{L}}$, $\vdash_S \varphi$ iff $\mathbb{K} \models \varphi \approx \bar{1}$. We will see in this section that MTL is in fact the weakest t-norm based logic.

T-norms had appeared in the framework of probabilistic metric spaces (see Schweizer and Sklar's works [133, 134]) following the ideas of Menger exposed in [112]. An extensive monography on t-norms can be found in [102].

Definition 3.5. A *t-norm* is a function $T : [0, 1]^2 \rightarrow [0, 1]$ such that for every $a, b, c \in [0, 1]$:

- $T(a, T(b, c)) = T(T(a, b), c)$ (associativity)
- $T(a, b) = T(b, a)$ (commutativity)
- If $b \leq c$, then $T(a, b) \leq T(a, c)$ (monotony)
- $T(a, 1) = a$ (neutral element)

Some properties follow immediately from the definition.

Proposition 3.6. *Let $T : [0, 1]^2 \rightarrow [0, 1]$ be a t-norm. For every $a, b \in [0, 1]$:*

- $T(a, 0) = 0$
- $T(a, b) \leq \min\{a, b\}$

Since they are binary functions we will often use an operational notation, such as $a * b$, instead of $T(a, b)$. Sometimes we will generalize slightly the notion of t-norm by considering also t-norms defined over other closed real intervals, satisfying exactly the same conditions.

The residuum of a t-norm is introduced in order to model the implication in fuzzy logics.

Definition 3.7. *Let $*$ be a t-norm. For every pair $\langle a, b \rangle \in [0, 1]^2$ we define the pseudocomplement of a with respect to b as: $a \rightarrow b := \sup\{c \in [0, 1] : a * c \leq b\}$.*

Proposition 3.8. *Let $*$ be a t-norm and consider the associated operation \rightarrow . $*$ and \rightarrow form an adjoint pair if, and only if, $*$ is left-continuous. In this case, \rightarrow is called the residuum of $*$.*

Proposition 3.9. *If $*$ is a left-continuous t-norm and \rightarrow is its residuum, then for every $a, b \in [0, 1]$ the following hold:*

- (i) $a \rightarrow b = \max\{c \in [0, 1] : a * c \leq b\}$.
- (ii) $a \rightarrow b = 1$ if, and only if, $a \leq b$.
- (iii) $(a \rightarrow b) \vee (b \rightarrow a) = 1$.
- (iv) $\max\{a, b\} = \min\{(a \rightarrow b) \rightarrow b, (b \rightarrow a) \rightarrow a\}$.

Therefore, given a left-continuous t-norm $*$, the algebra $[0, 1]_* = \langle [0, 1], *, \rightarrow, \min, \max, 0, 1 \rangle$ is an MTL-chain. Notice that $[0, 1]_*$ is completely determined by the t-norm. Moreover, it is obvious that in every MTL-chain \mathcal{A} over $[0, 1]$, the operation $\&^{\mathcal{A}}$ is a left-continuous t-norm. We call these chains defined over $[0, 1]$ *standard algebras*. Sometimes we will consider the isomorphic copy of a standard chain $[0, 1]_*$ over some other closed real interval $[a, b]$ (being the isomorphism the affine transformation from $[0, 1]$ to $[a, b]$), and we will denote it as $[a, b]_*$.

The standard BL-chains, are those standard MTL-chains where the t-norm and its residuum satisfy the divisibility condition, i.e. $\min\{a, b\} = a * (a \rightarrow b)$ for every $a, b \in [0, 1]$. It is well known the following characterization:

Proposition 3.10. *Let $[0, 1]_* = \langle [0, 1], *, \rightarrow, \min, \max, 0, 1 \rangle$ be a standard MTL-chain. $[0, 1]_*$ is a BL-algebra if, and only if, $*$ is a continuous t-norm.*

In general, a two-place function can be continuous in each argument without being continuous, but, thanks to the monotony, this is not the case of t-norms.

Proposition 3.11. *A t-norm is continuous if, and only if, it is continuous in each argument.*

Therefore, due to commutativity, the continuity of a t-norm is equivalent to the continuity in its first argument.

It is interesting to remark that the stated equivalence between continuity and divisibility in t-norms cannot be generalized to all MTL-chains. Actually, Boixader, Esteva and Godo have proved the following:

Proposition 3.12 ([23]). *In every BL-chain the monoidal operation is continuous with respect to the order topology.*

They give also a counterexample that shows that the inverse implication is not true, i.e. an MTL-chain where the monoidal operation is continuous with respect to the order topology and the divisibility does not hold. Nevertheless, the equivalence between continuity and divisibility can still be slightly generalized:

Proposition 3.13 ([23]). *Let \mathcal{A} be an MTL-chain whose order is dense and complete. Then $\&^{\mathcal{A}}$ is continuous if, and only if, \mathcal{A} is a BL-chain.*

We recall now the definitions of several well known examples of continuous t-norms and their associated standard BL-chains:

1. There is only one standard G-chain and it is the one defined by the minimum t-norm, i.e. the t-norm: $a *_G b = \min\{a, b\}$. The residuum is:

$$a \rightarrow_G b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

and the negation is:

$$\neg a = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{otherwise.} \end{cases}$$

which is called *Gödel negation*. Of course, $*_G$ is continuous. We denote this algebra by $[0, 1]_G$. Notice that all the elements are idempotent and that it satisfies the pseudocomplementation equation ($x \wedge \neg x \approx \bar{0}$).

2. All the standard Π -chains are isomorphic to the one defined by the product of real numbers: $a *_\Pi b = ab$. It is clearly a continuous t-norm as well, and its residuum is *Goguen implication*:

$$a \rightarrow_\Pi b = \begin{cases} 1 & \text{if } a \leq b, \\ b/a & \text{otherwise.} \end{cases}$$

and the negation is Gödel negation. The algebra is denoted by $[0, 1]_{\Pi}$. Notice that the only idempotent elements are 0 and 1 and the only nilpotent elements is 0. Moreover, it satisfies the pseudocomplementation equation and it is cancellative, i.e. for every $a, b, c \in [0, 1]$, if $c \neq 0$ and $a * c = b * c$, then $a = b$.

3. All the standard MV-chains are isomorphic to the one defined by Łukasiewicz t-norm: $a *_L b = \max\{0, a + b - 1\}$. Its residuum is:

$$x \rightarrow_L y = \begin{cases} 1 & \text{if } a \leq b, \\ 1 - a + b & \text{otherwise.} \end{cases}$$

and its negation is the standard involutive negation: $\neg a = 1 - a$. This t-norm is also continuous and, interestingly, even its residuum is continuous.

The associated standard algebra is denoted by $[0, 1]_L$. Notice that the only idempotent elements are 0 and 1. It satisfies the following weak form of cancellation: for every $a, b, c \in [0, 1]$, if $a * c = b * c \neq 0$, then $a = b$.

Theorem 3.14. *MTL, BL, L, Π and G are t-norm based logics. In fact, they enjoy the following standard completeness results:*

1. *For every formula $\varphi \in Fm_{\mathcal{L}}$ and every set of formulae $\Gamma \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\text{MTL}} \varphi$ iff $\{\psi \approx \bar{1} : \psi \in \Gamma\} \models_{[0,1]^*} \varphi \approx \bar{1}$ for every left-continuous t-norm $*$ ([100]).*
2. *For every formula $\varphi \in Fm_{\mathcal{L}}$ and every finite set of formulae $\Gamma \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\text{BL}} \varphi$ iff $\{\psi \approx \bar{1} : \psi \in \Gamma\} \models_{[0,1]^*} \varphi \approx \bar{1}$ for every continuous t-norm $*$ ([30]).*
3. *For every formula $\varphi \in Fm_{\mathcal{L}}$ and every finite set of formulae $\Gamma \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_L \varphi$ iff $\{\psi \approx \bar{1} : \psi \in \Gamma\} \models_{[0,1]_L} \varphi \approx \bar{1}$ ([86]).*
4. *For every formula $\varphi \in Fm_{\mathcal{L}}$ and every finite set of formulae $\Gamma \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\Pi} \varphi$ iff $\{\psi \approx \bar{1} : \psi \in \Gamma\} \models_{[0,1]_{\Pi}} \varphi \approx \bar{1}$ ([83]).*
5. *For every formula $\varphi \in Fm_{\mathcal{L}}$ and every set of formulae $\Gamma \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_G \varphi$ iff $\{\psi \approx \bar{1} : \psi \in \Gamma\} \models_{[0,1]_G} \varphi \approx \bar{1}$ ([47]).*

Therefore, it is now clear that L, Π and G are respectively the logics of the three main continuous t-norms, BL is the logic of all continuous t-norms and MTL is the logic of all left-continuous t-norms. Since a t-norm has a residuum if, and only if, is left-continuous, MTL is the weakest t-norm based fuzzy logic and thus, the basis for an investigation on such kind of logics.

Axiom schema	Name
$\neg\neg\varphi \rightarrow \varphi$	Involution (Inv)
$\neg\neg\chi \rightarrow ((\varphi \& \chi \rightarrow \psi \& \chi) \rightarrow (\varphi \rightarrow \psi))$	Cancellation (III)
$\varphi \rightarrow \varphi \& \varphi$	Contraction (Con)
$\varphi \wedge \psi \rightarrow \varphi \& (\varphi \rightarrow \psi)$	Divisibility (Div)
$\varphi \wedge \neg\varphi \rightarrow \bar{0}$	Pseudocomplementation (PC)
$\varphi \vee \neg\varphi$	Excluded Middle (EM)
$(\varphi \& \psi \rightarrow \bar{0}) \vee (\varphi \wedge \psi \rightarrow \varphi \& \psi)$	Weak Nilpotent Minimum (WNM)

Table 3.1: Some usual axiom schemata in fuzzy logics.

Logic	Additional axiom schemata
SMTL	(PC)
IIMTL	(PC) and (III)
IMTL	(Inv)
WNM	(WNM)
NM	(Inv) and (WNM)
BL	(Div)
SBL	(Div) and (PC)
L	(Div) and (Inv)
II	(Div), (PC) and (III)
G	(Con)
CPC	(EM)

Table 3.2: Some axiomatic extensions of MTL obtained by adding the corresponding additional axiom schemata.

3.4 Axiomatic extensions of MTL

The topic of the dissertation is the algebraic study of axiomatic extensions of MTL. As we have already mentioned, several well-known logics have been proved to be axiomatizable by adding some schemata to the Hilbert-style calculus of MTL. Besides, some other logics have been introduced in the literature by considering extensions of MTL with some relevant schemata. Table 3.1 and 3.2 collect some axiom schemata and the axiomatic extensions of MTL that they define. The partially ordered set defined by them is depicted in Figure 3.1.

We have already explained how BL, L, G and II appeared. IMTL was introduced in [51] as the involutive axiomatic extension of MTL, and it was conjectured to be the logic of involutive left-continuous t-norms, which was proved in [49]. NM and WNM logics were also introduced in [51]; the first one was intended to capture the logic of the only known example at that time of left-continuous non-continuous t-norm, $[0, 1]_{\text{NM}}$, while WNM was its non-involutive generalization. Standard completeness theorems for both NM and WNM were

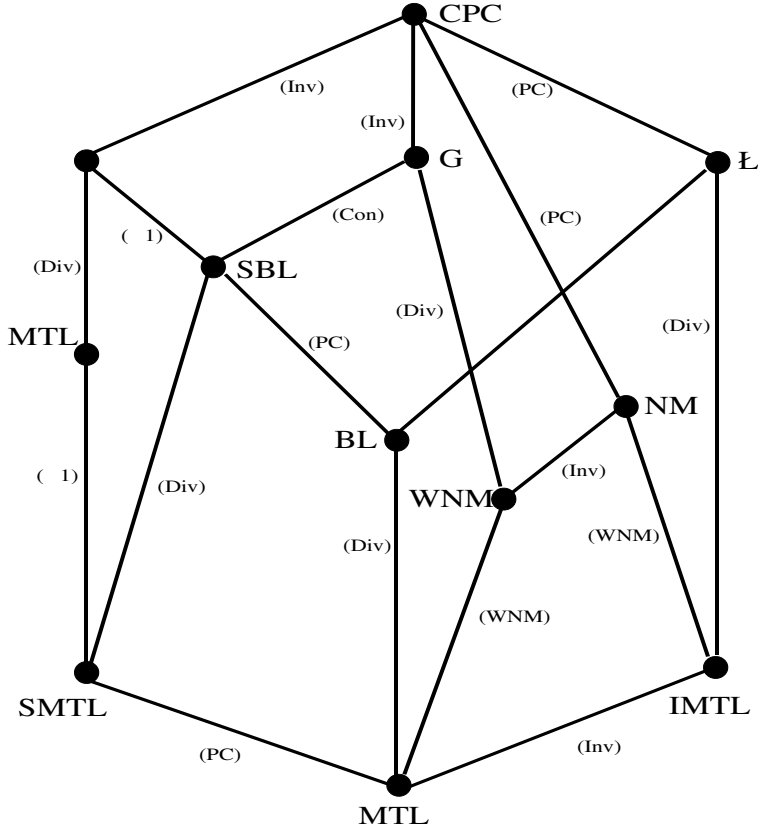


Figure 3.1: Graphic of some axiomatic extensions of MTL.

already proved in [51]. As for SBL and SMTL, they were defined respectively in [54] and [49] in order to capture the logic of continuous and left-continuous resp. t-norms with a Gödel negation; their standard completeness is proved in [30] and in [49] respectively. Finally, Π MTL was introduced in [81] as the cancellative extension of MTL (a non-divisible generalization of Π), and its standard completeness is proved in [88].

Chapter 4

Structure of MTL-algebras

The algebraization results presented in the previous chapter make clear that the study of t-norm based fuzzy logics and their axiomatic extensions strongly relies on the knowledge of MTL-algebras and their structure. In this chapter we prove the well-known decomposition of MTL-algebras as a subdirect product of MTL-chains. Therefore, a lot of general problems of MTL-algebras can be reduced to chains. In particular, the structure of BL-chains has been fully described by means of a generalization of the Ling and Mostert and Shields representation theorem for standard BL-chains. Unfortunately, a general representation theorem for MTL-chains is far from being known. Nevertheless, we will give some useful insight to the structure of MTL-chains by considering the notions of filter, Archimedean classes, ordinal sums of totally ordered semihoops, and some methods for constructing new classes of IMTL-chains (studied by Jenei in [97, 98, 99]). Many of the results that we will give in this chapter are proved or are straightforward from several foundational works in the subject; see for instance [51, 104, 87].

4.1 Basic definitions and results

The first basic notions are the filters and the implicative filters.

Definition 4.1. *Let \mathcal{A} be an MTL-algebra. A filter is a set $F \subseteq A$ such that:*

- $\bar{1}^A \in F$,
- If $a \in F$ and $a \leq b$, then $b \in F$, and
- If $a, b \in F$, then $a \& b \in F$.

A subset $F \subseteq A$ is an implicative filter iff it satisfies:

- $\bar{1}^A \in F$
- If $a, a \rightarrow b \in F$, then $b \in F$

These two notions coincide:

Proposition 4.2. *Let F be a subset of the carrier of an MTL-algebra. Then, F is an implicative filter if, and only if, F is a filter.*

Definition 4.3. *Let F be a filter of an MTL-algebra. F is proper iff $\bar{0}^A \notin F$. F is a prime filter iff F is proper and for every $a, b \in A$ if $a \vee b \in F$, then $a \in F$ or $b \in F$.*

Proposition 4.4. *The family of all filters of an MTL-algebra \mathcal{A} is a closure system, i.e. it is a family of subsets of A closed under arbitrary intersections and containing A .*

Therefore, it makes sense to speak about the notion of *generated filter*.

Definition 4.5. *Let \mathcal{A} be an MTL-algebra and $B \subseteq A$ an arbitrary subset. The filter generated by B is the minimum filter of \mathcal{A} containing B , i.e. the intersection of all filters containing B . We denote it by $\mathbb{F}i(B)$. When the filter is generated only by an element $a \in A$, we write F^a instead of $\mathbb{F}i(\{a\})$, and we call it a principal filter.*

We will use the following notations:

1. $\mathbb{F}i(\mathcal{A})$ denotes the set of proper filters of \mathcal{A} .
2. Given a filter $F \in \mathbb{F}i(\mathcal{A})$, \bar{F} denotes the set $\{a \in A : \neg a \in F\}$

Notice that if \mathcal{A} is an IMTL-algebra, then $\bar{\bar{F}} = \neg F = \{\neg a : a \in F\}$.

Definition 4.6. *If \mathcal{A} is an MTL-algebra and $a \in A$, we define $a^0 := \bar{1}^A$, $a^1 := a$ and for every natural number $n > 1$, $a^n := a^{n-1} * a$.*

Definition 4.7. *Given an MTL-algebra \mathcal{A} , $a \in A$ is idempotent if, and only if, $a^2 = a$. $\text{Id}(\mathcal{A})$ will be the set of all idempotent elements of \mathcal{A} . Notice that $\bar{0}^A, \bar{1}^A \in \text{Id}(\mathcal{A})$. $a \in A$ is nilpotent if, and only if, $a^n = \bar{0}^A$ for some $n \geq 1$.*

Proposition 4.8. *Let \mathcal{A} be an MTL-algebra and $B \subseteq A$ an arbitrary subset. Then the filter generated by B can be described as $\mathbb{F}i(B) = \{a \in A : b_1^{n_1} * \dots * b_k^{n_k} \leq a \text{ for some } k, n_1, \dots, n_k \geq 1 \text{ and some } b_1, \dots, b_k \in B\}$.*

The last proposition has as a consequence the following form of LDDT for MTL.

Theorem 4.9 ([51]). *For every $\Gamma \cup \{\varphi, \psi\} \subseteq \text{Fm}_{\mathcal{L}}$, $\Gamma \cup \{\varphi\} \vdash_{\text{MTL}} \psi$ if, and only if, there exists $n \geq 1$ such that $\Gamma \vdash_{\text{MTL}} \varphi^n \rightarrow \psi$, where φ^n denotes $\varphi \& \dots^n \& \varphi$.*

By Theorem 2.22 this implies that MTL, and hence all its subvarieties, enjoy the CEP.

There is also a one-to-one correspondence between filters and congruences.

Proposition 4.10. *Let \mathcal{A} be an MTL-algebra. For every filter $F \subseteq A$ we define $\Theta(F) := \{\langle a, b \rangle \in A^2 : a \leftrightarrow b \in F\}$, and for every congruence θ of \mathcal{A} we define $Fi(\theta) := \{a \in A : \langle a, 1 \rangle \in \theta\}$. Then, Θ is an order isomorphism from the set of filters onto the set of congruences and Fi is its inverse.*

By virtue of this correspondence, we will do a notational abuse by writing A/F instead of $A/\Theta(F)$. Given an element $a \in A$, $[a]_F$ will denote the equivalence class of a w.r.t. to the congruence $\Theta(F)$.

Given an MTL-algebra \mathcal{A} and an element $a \in A$, we say that a is the *fixpoint* of \mathcal{A} if, and only if, $a = \neg a$. In [87] is proved that there exists at most one fixpoint.¹

Definition 4.11. *Let \mathcal{A} be an MTL-algebra. The sets of positive and negative elements of \mathcal{A} are respectively defined as:*

$$\begin{aligned} A_+ &:= \{a \in A : a > \neg a\} \\ A_- &:= \{a \in A : a \leq \neg a\} \end{aligned}$$

Consider the terms $p(x) := x \vee \neg x$ and $n(x) := x \wedge \neg x$. The next proposition is an easy but useful result describing these sets.

Proposition 4.12. *Let \mathcal{A} be an MTL-algebra. Then:*

- $A_+ = \{p(a) : a \in A, \neg a \neq \neg\neg a\}.$
- $A_- = \{n(a) : a \in A\}.$

Notice that $p(a)$ is the fixpoint if, and only if, $\neg a = \neg\neg a$.

Definition 4.13. *Let \mathcal{A} be an MTL-algebra. An element $a \in A \setminus \{\bar{0}^A\}$ is a zero divisor if, and only if, there exists $b \in A \setminus \{\bar{0}^A\}$ such that $a \&^A b = \bar{0}^A$.*

The residuation implies this property: for every a, b in an MTL-algebra \mathcal{A} , $a \leq \neg^A b$ iff $a \&^A b = \bar{0}^A$. Therefore, all negative elements (except for $\bar{0}^A$) are zero divisors. The following proposition is also straightforward.

Proposition 4.14. *Let \mathcal{A} be an MTL-chain. The following are equivalent:*

- \mathcal{A} is an SMTL-chain.
- \mathcal{A} has no zero divisors.

Using Zorn's Lemma one can prove that for each proper filter F , there is a maximal proper filter G containing F . Moreover, every maximal filter is prime. $Max(\mathcal{A})$ will denote the set of all maximal filters. The radical of \mathcal{A} is defined as $Rad(\mathcal{A}) := \bigcap Max(\mathcal{A})$. Note that in a chain the set of filters is totally ordered, hence the radical is the maximum proper filter and $Rad(\mathcal{A}) \subseteq A_+$.

The following known characterization of the elements of a maximal filter will be useful.

¹Actually, Hhle states it for the involutive algebras, but the same proof gives the result for the general non-involutive case.

Proposition 4.15. *Let \mathcal{A} be an MTL-algebra and $M \subseteq A$ a maximal filter. Then for every $a \in A$, $a \notin M$ iff there is n such that $\neg a^n \in M$.*

The correspondence between filters and congruences entails immediately the following proposition.

Proposition 4.16. *Let \mathcal{A} be an MTL-algebra and $F \subseteq A$ a proper filter. The following are equivalent:*

- (i) \mathcal{A}/F is subdirectly irreducible.
- (ii) F is \cap -completely irreducible.
- (iii) F is maximal relatively to an element, i.e. there is an element $a \in A$ such that F is maximal in the set of proper filters not containing a .

In particular, when $F = \{\bar{1}^{\mathcal{A}}\}$ we obtain:

Corollary 4.17. *Let \mathcal{A} be an MTL-algebra. The following are equivalent:*

- (i) \mathcal{A} is subdirectly irreducible.
- (ii) $\{\bar{1}^{\mathcal{A}}\}$ is \cap -completely irreducible.
- (iii) \mathcal{A} has a minimum non-trivial filter.

More generally, the finitely subdirectly irreducible members of MTL are characterized by means of the following proposition.

Proposition 4.18 ([51]). *Let \mathcal{A} be an MTL-algebra and $F \subseteq A$ a proper filter. The following are equivalent:*

- (i) F is prime.
- (ii) \mathcal{A}/F is finitely subdirectly irreducible.
- (iii) \mathcal{A}/F is a chain.

Corollary 4.19 ([51]). *An MTL-algebra \mathcal{A} is finitely subdirectly irreducible if, and only if, for every $a, b \in A$ such that $a \vee b = \bar{1}^{\mathcal{A}}$, $a = \bar{1}^{\mathcal{A}}$ or $b = \bar{1}^{\mathcal{A}}$.*

Corollary 4.20 ([51]). *An MTL-algebra \mathcal{A} is finitely subdirectly irreducible if, and only if, it is a chain.*

Thus, we obtain the basic result of this section:

Theorem 4.21 ([51]). *Each MTL-algebra is representable as a subdirect product of MTL-chains.*

This decomposition result has its equivalent logical form:

Theorem 4.22 ([51]). *Given $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\text{MTL}} \varphi$ if, and only if, $\{\psi \approx \bar{1} : \psi \in \Gamma\} \models_{\{\text{MTL-chains}\}} \varphi \approx \bar{1}$.*

The same decomposition result is true for every axiomatic extension of MTL (i.e. for every subvariety of MTL). It can also be proved in the same way for the $\bar{0}$ -free subreducts of MTL-algebras. These algebras are called *prelinear semihoops* and they are defined as follows.

Definition 4.23 ([52]). *An algebra $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ of type $\langle 2, 2, 2, 0 \rangle$ is a prelinear semihoop² iff:*

- $\mathcal{A} = \langle A, \wedge^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ is an inf-semilattice with upper bound.
- $\langle A, \&^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ is a commutative monoid isotonic w.r.t. the inf-semilattice order.
- For every $a, b \in A$, $a \leq b$ iff $a \rightarrow b = \bar{1}^{\mathcal{A}}$.
- For every $a, b, c \in A$, $a \&^{\mathcal{A}} b \rightarrow^{\mathcal{A}} c = a \rightarrow^{\mathcal{A}} (b \rightarrow^{\mathcal{A}} c)$.

An operation $\vee^{\mathcal{A}}$ is defined as: $a \vee^{\mathcal{A}} b = ((a \rightarrow^{\mathcal{A}} b) \rightarrow^{\mathcal{A}} b) \wedge^{\mathcal{A}} ((b \rightarrow^{\mathcal{A}} a) \rightarrow^{\mathcal{A}} a)$. \mathcal{A} is called prelinear iff for every $a, b, c \in A$, $(a \rightarrow^{\mathcal{A}} b) \rightarrow^{\mathcal{A}} c \leq ((b \rightarrow^{\mathcal{A}} a) \rightarrow^{\mathcal{A}} c) \rightarrow^{\mathcal{A}} c$. If \mathcal{A} has a minimum element, then it is called a bounded prelinear semihoop (i.e. an MTL-algebra).

\mathcal{A} is a hoop iff $\mathcal{A} \models x \& (x \rightarrow y) \approx y \& (y \rightarrow x)$. A Wajsberg hoop is a hoop satisfying $(x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x$. A hoop \mathcal{A} is cancellative if $a \&^{\mathcal{A}} b \leq c \&^{\mathcal{A}} b$ implies $a \leq c$, for every $a, b, c \in A$. In [52] is proved that cancellative hoops coincide with unbounded Wajsberg hoops.

The class of all prelinear semihoops is a variety and, since $\bar{0}$ is not involved neither in the defining equation nor in the equivalence formulae, the $\bar{0}$ -free fragments of MTL and its axiomatic extensions are still algebraizable with the same defining equation nor in the equivalence formulae and the corresponding class of prelinear semihoops as equivalent algebraic semantics. Prelinear hoops are the $\bar{0}$ -free subreducts of BL-algebras, Wajsberg hoops are the $\bar{0}$ -free subreducts of MV-algebras and product hoops are the $\bar{0}$ -free subreducts of Π -algebras (see [58, 2, 3, 52]).³ It is interesting to notice that filters of MTL-algebras coincide with universes of prelinear semihoops.

We will finish the section with some final remarks on subdirectly irreducible MTL-algebras. It has been proved that they form a subclass of MTL-chains. Clearly it is proper subclass (just consider the chain $[0, 1]_{\mathbf{G}}$ where all the elements are idempotent and hence there is not a minimum non-trivial filter). This class has not been yet characterized by means of a nicer description, but we can easily prove the following sufficient condition.

Proposition 4.24. *If \mathcal{A} is an MTL-chain with a coatom, then it is subdirectly irreducible.*

²These algebras are sometimes also called MTLH-algebras.

³Although prelinear semihoops and prelinear hoops were originally called in the literature *basic semihoops* and *basic hoops*, respectively, we prefer to refer them as *prelinear* since prelinearity is their real defining property.

Proof: Let $a = \max A \setminus \{\bar{1}^A\}$ be the coatom. Then it is clear that F^a is the minimum non-trivial filter. \square

This condition is necessary. Indeed, $[0, 1]_L$ is sudirectly irreducible but it has no coatom.

An important subclass of subdirectly irreducible algebras is the class of simple algebras, i.e. those without non-trivial congruences. They admit the following general characterization.

Proposition 4.25. *Let \mathcal{A} be an MTL-algebra. \mathcal{A} is simple if, and only if, for every $a \in A \setminus \{\bar{1}^A\}$, there is $k \geq 1$ such that $a^k = \bar{0}^A$.*

4.2 Archimedean classes

A very useful notion to study the structure of MTL-chains is that of Archimedean class. It has been already used by Horčík in his Ph. D. dissertation ([89]) for the description of IIMTL chains, but some of his results are actually valid for all MTL-chains. In this section, we recall the definition of Archimedean class for MTL-chains and totally ordered semihoops and state some of their properties.

Definition 4.26. *Let \mathcal{A} be an MTL-chain or a totally ordered semihoop. We define a binary relation \sim on A by letting for every $a, b \in A$, $a \sim b$ if, and only if, there is $n \geq 1$ such that $a^n \leq b \leq a$ or $b^n \leq a \leq b$. It is easy to check that \sim is an equivalence relation. Its equivalence classes are called Archimedean classes. Given $a \in A$, its Archimedean class is denoted as $[a]_\sim$.*

The following lemma is straightforward.

Lemma 4.27. *Let \mathcal{A} be an MTL-chain or a totally ordered semihoop and let $a, b \in A$. Then:*

1. $[a]_\sim$ is closed under $\&$.
2. $[a]_\sim$ is a convex subset of A .
3. $[\bar{1}^A]_\sim = \{\bar{1}^A\}$.
4. If \mathcal{A} is an MTL-chain, then $A_- \subseteq [\bar{0}^A]_\sim$.
5. $[a \& b]_\sim = [a \wedge b]_\sim$.

The next lemma and proposition show that Archimedean classes and filters are related in an interesting way, as is described in [89] for the case of IIMTL-chains, and generalized to MTL-chains in [91].

Lemma 4.28 ([91]). *Let \mathcal{A} be an MTL-chain. Then:*

1. The set of filters of \mathcal{A} is closed under unions.

2. Every filter of \mathcal{A} is a union of principal filters.
3. If $\text{Con}(\mathcal{A})$ is finite, then every filter of \mathcal{A} is principal.
4. For every $a \in A$, the principal filter generated by a , F^a , has a predecessor in the set of all filters ordered by inclusion. It is denoted as F_a .

Proposition 4.29 ([91]). *Let \mathcal{A} be an MTL-chain. There is a dual order isomorphism Φ between the set of Archimedean classes of \mathcal{A} and the set of its principal filters. Actually, for every $a \in A$, $\Phi([a]_{\sim}) = F^a$, and $\Phi^{-1}(F^a) = F^a \setminus F_a$.*

Corollary 4.30 ([91]). *An MTL-chain has a finite number of Archimedean classes if, and only if, its lattice of congruences is finite.*

4.3 The radical of MTL-algebras

The radical has been a useful notion in the study of MV-algebras and BL-algebras. In [63] the following characterization of the radical is given for MV-algebras (in the equivalent form of Wajsberg algebras):

If \mathcal{A} is an MV-algebra, then $\text{Rad}(\mathcal{A}) = \{a \in A : a^n > \neg a \quad \forall n \geq 1\}$.

Moreover, the radical of BL-algebras has been studied by Sessa and Turunen in [135], obtaining this description:

If \mathcal{A} is a BL-algebra, then $\text{Rad}(\mathcal{A}) = \{a \in A : \neg\neg a^n > \neg a \quad \forall n \geq 1\}$.

Afterwards this result has been improved by Cignoli and Torrens in [33], obtaining:

If \mathcal{A} is a BL-algebra, then $\text{Rad}(\mathcal{A}) = \{a \in A : a^n > \neg a \quad \forall n \geq 1\}$,

i.e. the same expression as in the involutive case. However, the property of divisibility was used in the proofs of both characterizations for the radical of BL-algebras. So it was not obvious how to generalize this result to MTL-algebras. Here we present a new proof for the whole class of MTL-algebras.

First we do it for chains:

Lemma 4.31. *Let \mathcal{A} be an MTL-chain. Then,*

$$\text{Rad}(\mathcal{A}) = \{a \in A : a^n > \neg a \quad \forall n \geq 1\}.$$

Proof: If $a \in \text{Rad}(\mathcal{A})$, then for every $n \geq 1$, $a^n \in \text{Rad}(\mathcal{A}) \subseteq A_+$. Since $a^n \leq a$, we obtain $\neg a \leq \neg a^n < a^n$. Conversely, take $a \in A$ such that for every $n \geq 1$, $a^n > \neg a$. Then, in particular, for every n , $a^n \neq \bar{0}^{\mathcal{A}}$, so the filter generated by a , F^a , is proper. Thus, $a \in F^a \subseteq \text{Rad}(\mathcal{A})$, since the set of filters of \mathcal{A} is totally ordered. \square

In order to extend the characterization to all MTL-algebras we will need some previous lemmata.

Lemma 4.32. *Let \mathcal{A} be an MTL-algebra and F a maximal filter of \mathcal{A} . Then for any subalgebra $\mathcal{B} \subseteq \mathcal{A}$, $F \cap B$ is a maximal filter of \mathcal{B} .*

Proof: It is straightforward to check that $F \cap B$ is a filter of \mathcal{B} . It is proper because $\bar{0}^A \notin F$. Moreover, we know that for every $a \in A$, $a \notin F$ iff $\exists n \neg a^n \in F$. Therefore it is obvious that for every $a \in B$, $a \notin F \cap B$ iff $\exists n \neg a^n \in F \cap B$. Thus $F \cap B$ is also maximal. \square

Lemma 4.33. *Let \mathcal{A} be an MTL-algebra. Then for any subalgebra $\mathcal{B} \subseteq \mathcal{A}$, $\text{Max}(\mathcal{B}) = \{M \cap B : M \in \text{Max}(\mathcal{A})\}$. Therefore, $\text{Rad}(\mathcal{B}) = \text{Rad}(\mathcal{A}) \cap B$.*

Proof: We know by the previous lemma that for every $M \in \text{Max}(\mathcal{A})$, $M \cap B \in \text{Max}(\mathcal{B})$. Take $F \in \text{Max}(\mathcal{B})$. Then, by the CEP, there is a proper filter F' of \mathcal{A} such that $F = F' \cap B$. But then there is a maximal filter $M \in \text{Max}(\mathcal{A})$ containing F' , so $F' \cap B \subseteq M \cap B$. Hence, since F is maximal in \mathcal{B} , we obtain $F = M \cap B$. \square

Next we will describe some maximal filters in direct products. To this end we need some more notation. Given a set of MTL-algebras $\{\mathcal{A}_i : i \in I\}$, $\hat{a} \in \prod_{i \in I} A_i$, $k \in I$ and $b \in A_k$, we define $\sigma_k(\hat{a}, b) \in \prod_{i \in I} A_i$ by:

$$\sigma_k(\hat{a}, b)_i = \begin{cases} a_i & \text{if } i \neq k, \\ b & \text{if } i = k. \end{cases}$$

Lemma 4.34. *Let $\{\mathcal{A}_i : i \in I\}$ be a set of MTL-algebras and consider their direct product $\prod_{i \in I} \mathcal{A}_i$. Then, for every $k \in I$ and every $M_k \in \text{Max}(\mathcal{A}_k)$, the set $M_k \times \prod_{i \neq k} A_i$ is a maximal filter of $\prod_{i \in I} \mathcal{A}_i$.*

Proof: It is easy to check that $M_k \times \prod_{i \neq k} A_i$ is a proper filter of $\prod_{i \in I} \mathcal{A}_i$. Moreover, for every $\hat{a} \in \prod_{i \in I} A_i$, $\hat{a} \notin M_k \times \prod_{i \neq k} A_i$ iff $a_k \notin M_k$ iff $\exists n \neg a_k^n \in M_k$ iff $\exists n \neg \hat{a}^n \in M_k \times \prod_{i \neq k} A_i$. Thus, $M_k \times \prod_{i \neq k} A_i$ is a maximal filter. \square

Lemma 4.35. *Given any set of MTL-algebras $\{\mathcal{A}_i : i \in I\}$, $\text{Rad}(\prod_{i \in I} \mathcal{A}_i) = \prod_{i \in I} \text{Rad}(\mathcal{A}_i)$.*

Proof: By applying the definition of the radical and the previous lemma we obtain:

$$\text{Rad}(\prod_{i \in I} \mathcal{A}_i) = \bigcap \text{Max}(\prod_{i \in I} \mathcal{A}_i) \subseteq \bigcap \{M_k \times \prod_{i \neq k} A_i : k \in I, M_k \in \text{Max}(\mathcal{A}_k)\} = \prod_{i \in I} \bigcap \text{Max}(\mathcal{A}_i) = \prod_{i \in I} \text{Rad}(\mathcal{A}_i).$$

Conversely, take $\hat{a} \in \prod_{i \in I} \text{Rad}(\mathcal{A}_i)$ and $M \in \text{Max}(\prod_{i \in I} \mathcal{A}_i)$. We must prove that $\hat{a} \in M$. Suppose not. Then, by the maximality of M , there is $\hat{m} \in M$ such that $\hat{a} \& \hat{m} = \hat{0}$, i.e. $a_i \& m_i = \bar{0}^{A_i}$, for every $i \in I$. Therefore, $m_i \leq \neg a_i$, for every $i \in I$. Since each $a_i \in \text{Rad}(\mathcal{A}_i)$, this implies $m_i \in (A_i)_-$, for every $i \in I$, so $\hat{m}^2 = \hat{0}$, contradicting $\hat{m} \in M$. \square

Theorem 4.36. *Let \mathcal{A} be an MTL-algebra. Then:*

$$\text{Rad}(\mathcal{A}) = \{a \in A : a^n > \neg a \quad \forall n \geq 1\}.$$

Proof: \mathcal{A} is representable as a subdirect product of some set of MTL-chains $\{\mathcal{A}_i : i \in I\}$. Using the previous lemmata we can compute the radical of \mathcal{A} in following way:

$$\text{Rad}(\mathcal{A}) = \text{Rad}(\prod_{i \in I} \mathcal{A}_i) \cap A = \prod_{i \in I} \text{Rad}(\mathcal{A}_i) \cap A = \prod_{i \in I} \{a_i \in A_i : a_i^n > \neg a_i \quad \forall n \geq 1\} \cap A = \{a \in A : a^n > \neg a \quad \forall n \geq 1\}. \quad \square$$

Corollary 4.37. *For every MTL-algebra \mathcal{A} , $\text{Rad}(\mathcal{A}) \subseteq A_+$.*

Corollary 4.38. *Let \mathcal{A} be an MTL-algebra. Then:*

A_+ is a filter if, and only if, $A_+ = \text{Rad}(\mathcal{A})$.

Proof: Suppose that A_+ is a filter. We have to prove that $A_+ \subseteq \text{Rad}(\mathcal{A})$. Let $a \in A_+$, then for every $n \geq 1$ $a^n \in A_+$. We have $a^n \leq a$, hence $\neg a \leq \neg a^n < a^n \leq a$ and this means that $a \in \text{Rad}(\mathcal{A})$. \square

4.4 Standard chains

Standard MTL-chains have been already introduced in the previous chapter, with some of their main properties. Since they form the intended semantics for the logics that we study, a closer look to their structure is necessary.

Given a standard MTL-chain $[0, 1]_* = \langle [0, 1], *, \rightarrow, \min, \max, 0, 1 \rangle$ determined by a left-continuous t-norm $*$, we can consider its negation operation which, as in every MTL-algebra, is defined as: $n(a) = a \rightarrow 0$, for each $a \in [0, 1]$. These negation operations coincide with the so-called *weak negation functions* studied in [136, 48].

Definition 4.39 ([48]). *A function $n : [0, 1] \rightarrow [0, 1]$ is a weak negation function if, and only if, it satisfies the following conditions:*

1. $n(1) = 0$.
2. If $a \leq b$, then $n(b) \leq n(a)$.
3. $a \leq n(n(a))$, for every $a \in [0, 1]$.

Proposition 4.40. *A function $n : [0, 1] \rightarrow [0, 1]$ is the negation of a standard MTL-chain if, and only if, it is a weak negation function.*

Definition 4.41 ([48]). *Two weak negation functions n_1 and n_2 are said to be isomorphic if, and only if, there is an increasing homeomorphism $f : [0, 1] \rightarrow [0, 1]$ such that $f(n_1(a)) = n_2(f(a))$ for every $a \in [0, 1]$.*

Definition 4.42 ([48]). *A weak negation function n is called involutive or strong negation if, and only if $a = n(n(a))$ for every $a \in [0, 1]$.*

All the involutive negations are strictly increasing bijections of $[0, 1]$ and are isomorphic to the standard strong negation: $n(a) = 1 - a$. Therefore, strong negations functions are continuous. However, weak negation functions in general are only left-continuous.

A weak negation function has a kind of symmetry; roughly speaking: if we complete its graphic by drawing vertical lines in the jumps, then the obtained graphic is symmetric with respect to the diagonal $x = y$. Formally:

Definition 4.43. *A decreasing function $n : [0, 1] \rightarrow [0, 1]$ is called symmetric with respect to the diagonal if, and only if, it satisfies the following conditions:*

1. If $a \in n([0, 1])$ and $n(a) = b$, then $a = n(b)$.
2. If $a \notin n([0, 1])$, then:
 - (i) n is constant in the interval $[a, n(n(a))]$ with value $n(a)$, and
 - (ii) for every $b > n(a)$ we have $n(b) < a$, i.e. $n(a)$ is a discontinuity on the right with $n(n(a)^-) = n(n(a))$ and $n(n(a)^+) = \inf\{c : n \text{ is constant in } [c, n(n(a))]\} < a$.

Proposition 4.44 ([48]). $n : [0, 1] \rightarrow [0, 1]$ is a weak negation function if, and only if, it is decreasing, symmetric with respect to the diagonal and $n(1) = 0$.

The standard WNM-chains are defined by a weak negation function n and a t-norm of the form:

$$a *_n b = \begin{cases} \min\{a, b\} & \text{if } a > n(b), \\ 0 & \text{otherwise.} \end{cases}$$

The residuum is:

$$a \rightarrow_n b = \begin{cases} 1 & \text{if } a \leq b, \\ \max\{n(a), b\} & \text{otherwise.} \end{cases}$$

These t-norms are left-continuous but not continuous in general. Notice also that this family includes $[0, 1]_G$ and all standard NM-chains. Actually, there is only one standard NM-chain up to isomorphism: the one defined by the standard involutive negation. We denote it by $[0, 1]_{NM}$ and its t-norm was first introduced by Fodor in [59].

The minimum, the product and Łukasiewicz t-norms are the most prominent examples of continuous t-norms (we will sometimes refer to them as *the three basic continuous t-norms*) because it is possible to describe all continuous t-norms in terms of these three distinguished ones by using the notion of ordinal sum. This notion was born in the field of ordered semigroups (see [39, 40, 64]). In the particular case of t-norms, the definition is the following:

Definition 4.45. Let $\{[a_i, b_i] : i \in I\}$ be a countable family of closed subintervals of $[0, 1]$ such that their interiors are pairwise disjoint. For every $i \in I$, let $*_i$ be a t-norm defined on $[a_i, b_i]^2$. The ordinal sum of this family of t-norms is the operation defined as:

$$x * y = \begin{cases} x *_m y & \text{if } \exists m \in I \text{ such that } \langle x, y \rangle \in [a_m, b_m]^2, \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

It is straightforward to prove that the ordinal sum of a family of continuous t-norms is a continuous t-norm, and the ordinal sum of a family of left-continuous t-norms is a left-continuous t-norm. In the following we sketch how (independently) Mostert and Shields ([118]) and Ling ([108]) obtained a description of continuous-t-norms in terms of ordinal sums of the three basic ones.

Proposition 4.46. *Let $*$ be a continuous t-norm and $u \in [0, 1]$ an idempotent element. Then, for every $a, b \in [0, 1]$ such that $a \leq u \leq b$, we have $a * b = a$. Therefore, the restrictions of $*$ to $[0, u]^2$ and to $[u, 1]^2$ are isomorphic to continuous t-norms and $*$ is its ordinal sum.*

Theorem 4.47 ([118, 108]). *Let $*$ be a continuous t-norm. Then, the set of its idempotent elements is a closed subset of $[0, 1]$, and thus, its complement is the union of a countable family of pairwise disjoint open intervals. Moreover, if $\mathcal{I}(\ast)$ denotes the family of the closures of these intervals, then:*

- (i) *For every interval $I \in \mathcal{I}(\ast)$, the restriction of $*$ to I^2 is isomorphic to $[0, 1]_{\Pi}$ if it does not have any nilpotent element, or to $[0, 1]_L$ otherwise.*
- (ii) *If $a, b \in [0, 1]$ and there is no $I \in \mathcal{I}(\ast)$ such that $a, b \in I$, then $a * b = \min\{a, b\}$.*

Thus, every continuous t-norm can be decomposed as an ordinal sum of the three basic ones.

4.5 Decomposition in ordinal sums of semihoops

In the previous section we have presented the well-known result of Ling and Mosterd and Shields that gives the decomposition of every continuous t-norm as ordinal sum of the three basic ones. Although this result is very useful for many purposes, it has a little problem: the t-norms that appear in the decomposition do not form a subalgebra of the original one (they do not share the top and the bottom element). This disadvantage has been overcome by using the notion of ordinal sum of hoops (first introduced in [58] for ordinal sums of two hoops and then generalized in [3] to sums of arbitrary families of hoops), which has resulted in a deeper knowledge on the structure of all BL-chains, not only the standard ones. Here we generalize it to our framework of MTL-chains by considering ordinal sums of totally ordered semihoops.

Definition 4.48. *Let $\langle I, \leq \rangle$ be a totally ordered set. For all $i \in I$, let \mathcal{A}_i be a totally ordered semihoop such that for $i \neq j$, $A_i \cap A_j = \{\bar{1}\}$. Then $\bigoplus_{i \in I} \mathcal{A}_i$ (the ordinal sum of the family $\{\mathcal{A}_i : i \in I\}$) is the structure whose universe is $\bigcup_{i \in I} A_i$ and whose operations are:*

$$x \& y = \begin{cases} x \&^{\mathcal{A}_i} y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \setminus \{\bar{1}\} \text{ with } i > j, \\ x & \text{if } x \in A_i \setminus \{\bar{1}\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

$$x \rightarrow y = \begin{cases} x \rightarrow^{\mathcal{A}_i} y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \text{ with } i > j, \\ \bar{1} & \text{if } x \in A_i \setminus \{\bar{1}\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

For every $i \in I$, \mathcal{A}_i is called a component of the ordinal sum.

If in addition I has a minimum, say i_0 , and \mathcal{A}_{i_0} is bounded, then the ordinal sum $\bigoplus_{i \in I} \mathcal{A}_i$ forms an MTL-chain.

Definition 4.49. A totally ordered semihoop is indecomposable if, and only if, it is not isomorphic to any ordinal sum of two non-trivial totally ordered semihoops.

Theorem 4.50 ([3]). Every totally ordered hoop (BL-algebra) is the ordinal sum of a family of Wajsberg hoops (whose first component is an MV-algebra).

Theorem 4.51 ([3]). Let \mathcal{A} be a totally ordered basic hoop (BL-algebra). The following are equivalent:

1. \mathcal{A} is indecomposable.
2. \mathcal{A} is a Wajsberg hoop (MV-algebra).

Therefore, all BL-chains are decomposable as ordinal of indecomposable pre-linear hoops, i.e. Wajsberg hoops. Moreover, in the case of standard BL-chains, this decomposition is given by the Archimedean classes. Unfortunately, this is not true in general. For instance, Chang's MV-algebra \mathcal{C} is indecomposable, but it has two non-trivial Archimedean components.

Using ordinal sums one can also describe the structure of all finite subdirectly irreducible elements (that is all chains) of the variety generated by a standard BL-algebra:

Theorem 4.52 ([3]). Let $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ be a sequence of the basic components and let $[0, 1]_* = \bigoplus_{i=1}^n \mathcal{A}_i$ be their ordinal sum. Then, the class of finitely subdirectly irreducible members of $\mathbf{V}([0, 1]_*)$ is $\mathbf{HSP}_U([0, 1]_*) = \mathbf{HSP}_U(\mathcal{A}_1) \cup (\mathbf{ISP}_U(\mathcal{A}_1) \oplus \mathcal{HSP}_U(\mathcal{A}_2)) \cup \dots \cup (\mathbf{ISP}_U(\mathcal{A}_1) \oplus \bigoplus_{i=2}^{n-1} \mathcal{ISP}_U(\mathcal{A}_i) \oplus \mathcal{HSP}_U(\mathcal{A}_n))$, where $\mathbf{H}, \mathbf{I}, \mathbf{S}, \mathbf{P}_U$ denote the operators homomorphic images, isomorphic images, subalgebras and ultraproducts in the language of BL-algebras while $\mathcal{H}, \mathcal{I}, \mathcal{S}, \mathcal{P}_U$ denote the same operators in the language of Basic hoops (that is on the $\bar{0}$ -free language).

Moreover, since for the three basic cases we have that $\mathbf{ISP}_U([0, 1]_{\mathbf{L}})$, $\mathbf{ISP}_U([0, 1]_{\Pi})$, $\mathbf{ISP}_U([0, 1]_{\mathbf{G}})$ are respectively all MV-chains, all product chains and all Gödel chains [44, 58, 47], it follows that $\mathcal{ISP}_U([0, 1]_{\mathbf{L}})$, $\mathcal{ISP}_U([0, 1]_{\Pi})$, $\mathcal{ISP}_U([0, 1]_{\mathbf{G}})$ are all $\bar{0}$ -free subreducts of MV-chains, product chains and Gödel chains. Therefore, from the above Theorem 4.52 and the characterization of varieties generated by a standard BL-algebra in [53], it follows that we can generalize those results to any standard BL-algebra.

Corollary 4.53. Let $[0, 1]_*$ be a standard BL-algebra. Then the class of all finite subdirectly irreducible algebras in $\mathbf{V}([0, 1]_*)$ is $\mathbf{ISP}_U([0, 1]_*)$.

Now we generalize the decomposition theorem to all MTL-chains.

Theorem 4.54. For every MTL-chain \mathcal{A} , there is a maximum decomposition as ordinal sum of indecomposable totally ordered semihoops, with the first one bounded.

Proof: First we need to define the set \mathcal{D} of decompositions of \mathcal{A} . For every $F \subseteq \mathcal{P}(A \setminus \{\bar{1}^{\mathcal{A}}\})$, $F \in \mathcal{D}$ if, and only if, F is a partition of $A \setminus \{\bar{1}^{\mathcal{A}}\}$ such that for every $B \in F$, $B \cup \{\bar{1}^{\mathcal{A}}\}$ is a subuniverse of the $\bar{0}$ -free reduct of \mathcal{A} (hence the universe of a totally ordered semihoop \mathcal{B}) and $\mathcal{A} = \bigoplus\{\mathcal{B} : B \in F\}$. A partial order \preceq is defined in \mathcal{D} in following way:

for every $F, G \in \mathcal{D}$, $F \preceq G$ if, and only if, for each $B \in G$ there is a $B' \in F$ such that $B \subseteq B'$, i.e. the decomposition G is finer than F .

We will use Zorn's Lemma to show that the partially ordered set $\langle \mathcal{D}, \preceq \rangle$ has some maximal element. Suppose that $\mathcal{C} = \{D_k : k \in K\}$ is a chain of $\langle \mathcal{D}, \preceq \rangle$. Then, we define the following equivalence relation on $A \setminus \{\bar{1}^{\mathcal{A}}\}$:

For every $a, b \in A \setminus \{\bar{1}^{\mathcal{A}}\}$, $a \equiv b$ if, and only if, a and b belong to the same class of D_k for every $k \in K$. Let $[a]_{\equiv}$ denote the equivalence class of a w.r.t \equiv . We will prove that $\{[a]_{\equiv} : a \in A \setminus \{\bar{1}^{\mathcal{A}}\}\} \in \mathcal{D}$ and it is an upper bound of \mathcal{C} . Take $a \in A \setminus \{\bar{1}^{\mathcal{A}}\}$. It is straightforward to check that $[a]_{\equiv} \cup \{\bar{1}^{\mathcal{A}}\}$ is closed under $\&$ and \rightarrow . Now take $a, b \in A \setminus \{\bar{1}^{\mathcal{A}}\}$ such that $a < b$ and $[a]_{\equiv} \neq [b]_{\equiv}$. Then, there is some $k \in K$ such that a and b are not in the same component of D_k , thus $a \& b = a$. Therefore, $\{[a]_{\equiv} : a \in A \setminus \{\bar{1}^{\mathcal{A}}\}\} \in \mathcal{D}$. Now take arbitrary $k \in K$ and $a \in A \setminus \{\bar{1}^{\mathcal{A}}\}$. Then, by the definition of \equiv all the elements of $[a]_{\equiv}$ must be in the same component of D_k , so $D_k \preceq \{[a]_{\equiv} : a \in A \setminus \{\bar{1}^{\mathcal{A}}\}\}$.

Therefore, by Zorn's Lemma for every $F \in \mathcal{D}$, there exists a maximal decomposition $M \in \mathcal{D}$ such that $F \preceq M$. Finally, we will prove that there is a maximum one, i.e. there cannot be two different maximal decompositions. Suppose that $M_1, M_2 \in \mathcal{D}$ are two different maximal elements. Then there is $A \in M_1$ which is not included in any element of M_2 . Moreover, A is indecomposable so it is not a union of elements of M_2 , thus there is $B \in M_2$ such that $A \cap B \neq \emptyset$ and $B \not\subseteq A$. Then it is easy to see that A could be decomposed as ordinal sum of $A \cap B$ and $A \setminus B$, a contradiction. \square

Corollary 4.55. *Let \mathcal{A} be an MTL-chain. If the partition $\{[a]_{\sim} : a \in A \setminus \{\bar{1}^{\mathcal{A}}\}\}$ given by the Archimedean classes gives a decomposition as ordinal sum, then it is the maximum one.*

Proof: With the notation of the previous proof, take an arbitrary $F \in \mathcal{D}$. For every $a \in A \setminus \{\bar{1}^{\mathcal{A}}\}$, there is some $B \in F$ such that $[a]_{\sim} \subseteq B$, since the elements of F are closed under $\&$. Therefore, $F \preceq \{[a]_{\sim} : a \in A \setminus \{\bar{1}^{\mathcal{A}}\}\}$. So if $\{[a]_{\sim} : a \in A \setminus \{\bar{1}^{\mathcal{A}}\}\} \in \mathcal{D}$, then it is the maximum. \square

Definition 4.56. *An MTL-chain is totally decomposable if, and only if, the partition given by its Archimedean classes gives a decomposition as ordinal sum.*

Corollary 4.57. *Each standard BL-chain is totally decomposable.*

The decomposition of each MTL-chain as an ordinal sum of indecomposable totally ordered semihoops seems to give some representation theorem for MTL-chains. Unfortunately, the class of indecomposable totally ordered semihoops

is really big. For instance, as the following proposition proves, all involutive MTL-chains are indecomposable.

Proposition 4.58. *All IMTL-chains are indecomposable.*

Proof: Let \mathcal{A} be an IMTL-chain. If $\mathcal{A} \cong \mathcal{B}_2$, it is clearly indecomposable. Suppose that $\mathcal{A} \not\cong \mathcal{B}_2$ and it is decomposable as ordinal sum of two non-trivial totally ordered semihoops, i.e. $\mathcal{A} \cong \mathcal{C}_1 \oplus \mathcal{C}_2$. Then, there is $a \in \mathcal{C}_2 \setminus \{\bar{1}^{\mathcal{A}}\}$ and it satisfies $\neg a = \bar{1}^{\mathcal{A}}$, but this contradicts the fact that the negation is involutive. \square

4.6 Jenei's methods: rotation and rotation-annihilation

In the development of Fuzzy Logic the continuous t-norms have played an important role for the reasons already mentioned in this dissertation. While they were already well studied and classified still few examples of left-continuous and non-continuous t-norms were known, although left-continuity had been proved to be the necessary and sufficient condition for a t-norm to have a residuum and, thus, to provide a suitable semantics for fuzzy logics. The first known example of a left-continuous non-continuous t-norm was the Nilpotent Minimum t-norm given by Fodor in [59], which defined the MTL-algebra that we have denoted $[0, 1]_{\text{NM}}$. It was defined by considering the minimum t-norm and "annihilating" it with the standard involutive negation $n(x) = 1 - x$, i.e. defining the t-norm to be 0 under the graph of such negation function. This construction was later on generalized in [51] to define the class of Weak Nilpotent Minimum t-norms, by considering the annihilation of the minimum by any weak negation function. It was further generalized in [96] where Jenei studied the annihilation of any continuous t-norm by an involutive negation and, finally, in [29] where the authors studied annihilations of any continuous t-norm by a weak negation function.

In addition, some other methods for obtaining involutive left-continuous non-continuous t-norms have been introduced in the last years by Jenei in [98, 99]. Actually, these methods provide not only standard algebras, but also IMTL-algebras in general. First we present the disconnected and connected rotation constructions.

Definition 4.59. *Let \mathcal{A} be a prelinear semihoop. The disconnected rotation of \mathcal{A} is an algebra denoted \mathcal{A}^* and defined as follows. Let $A \times \{0\}$ be a disjoint copy of A . For every $a \in A$ we write a' instead of $\langle a, 0 \rangle$. Consider $\langle A' = \{a' : a \in A\}, \leq \rangle$ with the inverse order and let $A^* := A \cup A'$. We extend these orderings to an order in A^* by putting $a' < b$ for every $a, b \in A$. Finally, we take the following operations in \mathcal{A}^* :*

$\bar{1}^{\mathcal{A}^*} := \bar{1}^{\mathcal{A}}$, $\bar{0}^{\mathcal{A}^*} := (\bar{1}^{\mathcal{A}})'$, $\wedge^{\mathcal{A}^*}$ the minimum w.r.t. the ordering, $\vee^{\mathcal{A}^*}$ the maximum w.r.t. the ordering,

$$\neg^{\mathcal{A}^*} a := \begin{cases} a' & \text{if } a \in A \\ b & \text{if } a = b' \in A' \end{cases}$$

$$a \&^{\mathcal{A}^*} b := \begin{cases} a \&^{\mathcal{A}} b & \text{if } a, b \in A \\ \neg^{\mathcal{A}^*} (a \rightarrow^{\mathcal{A}} \neg^{\mathcal{A}^*} b) & \text{if } a \in A, b \in A' \\ \neg^{\mathcal{A}^*} (b \rightarrow^{\mathcal{A}} \neg^{\mathcal{A}^*} a) & \text{if } a \in A', b \in A \\ \bar{0}^{\mathcal{A}^*} & \text{if } a, b \in A' \end{cases}$$

$$a \rightarrow^{\mathcal{A}^*} b := \begin{cases} a \rightarrow^{\mathcal{A}} b & \text{if } a, b \in A \\ \neg^{\mathcal{A}^*} (a \&^{\mathcal{A}} \neg^{\mathcal{A}^*} b) & \text{if } a \in A, b \in A' \\ \bar{1}^{\mathcal{A}^*} & \text{if } a \in A', b \in A \\ \neg^{\mathcal{A}^*} b \rightarrow^{\mathcal{A}} \neg^{\mathcal{A}^*} a & \text{if } a, b \in A' \end{cases}$$

It is clear from this definition that Chang's algebra \mathcal{C} introduced in the previous chapter is the disconnected rotation of the cancellative hoop subreduct generated by $\frac{1}{2}$ in $[0, 1]_{\Pi}$.

Definition 4.60. Let \mathcal{A} be an MTL-algebra satisfying one of the following conditions:

- \mathcal{A} does not have zero divisors.
- $\exists c \in A$ such that $\forall a \in A$ zero divisor, $\neg a = c$.

Then, the connected rotation of \mathcal{A} is denoted \mathcal{A}^* and defined as follows.

Take $\langle A' = \{a' : a \in A, a \neq \bar{0}^{\mathcal{A}}\}, \leq \rangle$, a disjoint copy of $A \setminus \{\bar{0}^{\mathcal{A}}\}$ with the inverse order, and define $\neg^{\mathcal{A}^*} \bar{0}^{\mathcal{A}} := \bar{0}^{\mathcal{A}}$ and all the operations as in the disconnected rotation.

As an example, one can check that the standard NM-chain $[0, 1]_{\text{NM}}$ is isomorphic to the connected rotation of $[0, 1]_{\text{G}}$.

Proposition 4.61 ([97]). Disconnected rotations are IMTL-algebras without fixpoint and connected rotations are IMTL-algebras with fixpoint.

Jenei also introduced the disconnected and connected rotation-annihilation constructions to produce new kinds of IMTL-algebras.

Definition 4.62. Let \mathcal{A} be a prelinear semihoop and \mathcal{B} be an IMTL-algebra such that $A \cap B = \emptyset$. An algebra \mathcal{C} , the disconnected rotation-annihilation of \mathcal{A} and \mathcal{B} , is defined as follows. Let $\langle A' = \{a' : a \in A\}, \leq \rangle$ be a disjoint copy of A as above (disjoint also with B) endowed with inverse ordering and let $C := A \cup A' \cup B$. The orderings are extended to C by letting $a' < b$ and $b < c$ for every $a, c \in A$ and every $b \in B$. Let $C^+ := A$, $C^0 := B$ and $C^- := A'$. Finally, the following operations are defined on \mathcal{C} :

$\bar{1}^{\mathcal{C}} := \bar{1}^{\mathcal{A}}$, $\bar{0}^{\mathcal{C}} := (\bar{1}^{\mathcal{A}})'$, $\wedge^{\mathcal{C}}$ the minimum w. r. t. the order, $\vee^{\mathcal{C}}$ the maximum w. r. t. the order,

$$\neg^{\mathcal{C}} a := \begin{cases} a' & \text{if } a \in A \\ b & \text{if } a = b' \in A' \\ \neg^{\mathcal{B}} a & \text{if } a \in B \end{cases}$$

$$a \&^C b := \begin{cases} a \&^{\mathcal{A}} b & \text{if } a, b \in C^+ \\ \neg^C(a \rightarrow^{\mathcal{A}} \neg^C b) & \text{if } a \in C^+, b \in C^- \\ \neg^C(b \rightarrow^{\mathcal{A}} \neg^C a) & \text{if } a \in C^-, b \in C^+ \\ \bar{0}^C & \text{if } a, b \in C^- \\ \bar{0}^C & \text{if } a, b \in C^0 \text{ and } a \leq \neg^C b \\ a \&^{\mathcal{A}_1} b & \text{if } a, b \in C^0 \text{ and } a \not\leq \neg^C b \\ b & \text{if } a \in C^+, b \in C^0 \\ a & \text{if } a \in C^0, b \in C^+ \\ \bar{0}^C & \text{if } a \in C^-, b \in C^0 \\ \bar{0}^C & \text{if } a \in C^0, b \in C^- \end{cases}$$

$$a \rightarrow^C b := \begin{cases} a \rightarrow^{\mathcal{A}} b & \text{if } a, b \in C^+ \\ \neg^C(a *^{\mathcal{A}} \neg^C b) & \text{if } a \in C^+, b \in C^- \\ \bar{1}^C & \text{if } a \in C^-, b \in C^+ \\ \neg^C b \rightarrow^{\mathcal{A}} \neg^C a & \text{if } a, b \in C^- \\ a \rightarrow^{\mathcal{A}_1} b & \text{if } a, b \in C^0 \text{ and } a \not\leq b \\ \bar{1}^C & \text{if } a, b \in C^0 \text{ and } a \leq b \\ b & \text{if } a \in C^+, b \in C^0 \\ \bar{1}^C & \text{if } a \in C^0, b \in C^+ \\ \bar{1}^C & \text{if } a \in C^-, b \in C^0 \\ \neg^C a & \text{if } a \in C^0, b \in C^- \end{cases}$$

Definition 4.63. Let \mathcal{A} be a prelinear semihoop and \mathcal{B} be an IMTL-algebra such that $A \cap B = \emptyset$. An algebra \mathcal{C} , the connected rotation-annihilation of \mathcal{A} and \mathcal{B} , is defined as follows. Let $\langle A' = \{a' : a \in A\}, \leq \rangle$ be a disjoint copy of A as above (disjoint also with B) endowed with inverse ordering and let $C := A \cup A' \cup B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\}$. We extend the orderings to C by letting $a' < b$ and $b < c$ for every $a, c \in A$ and every $b \in B$. Let $C^+ := A$, $C^0 := B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\}$ and $C^- := A'$. Finally, the operations \mathcal{C} are defined as in the disconnected rotation-annihilation. We will denote \mathcal{C} as $\mathcal{A} \odot \mathcal{B}$.

Proposition 4.64 ([99]). Let \mathcal{A} and \mathcal{B} be a prelinear semihoop and an IMTL-algebra respectively. Then, the disconnected and connected rotation-annihilations of \mathcal{A} and \mathcal{B} are IMTL-algebras.

It is clear that in every connected rotation-annihilation $\mathcal{A} \odot \mathcal{B}$ the set A is a proper filter of $\mathcal{A} \odot \mathcal{B}$. Moreover, by using again Zorn's Lemma, we can prove that every IMTL-chain has a maximum decomposition as a connected rotation-annihilation.

Proposition 4.65. Let \mathcal{A} be an IMTL-chain. Then, there is a maximum proper filter F of \mathcal{A} such that $\mathcal{A} \cong \mathcal{C} \odot \mathcal{B}$, where \mathcal{C} is the semihoop determined by A and \mathcal{B} is the IMTL-chain determined by $(A \setminus (F \cup \neg F)) \cup \{\bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}}\}$.

Notice that in the previous proposition, \mathcal{B} is simple if, and only if, $F = \text{Rad}(\mathcal{A})$.

Proposition 4.66. *Given an IMTL-chain \mathcal{A} , a proper filter $F \in Fi(\mathcal{A})$ and $B := (A \setminus (F \cup \neg F)) \cup \{\bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}}\}$, the following are equivalent:*

- (i) *F and B give a decomposition of \mathcal{A} as a connected rotation-annihilation.*
- (ii) *B is a subuniverse and $\mathcal{A}/F \cong \mathcal{B}$.*
- (iii) *For each $a \in F$ and $b \in A_+ \setminus F$, $a \& b = b$.*
- (iv) *B is a subuniverse and $B \setminus \{\bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}}\}$ is convex w.r.t. the order of \mathcal{A} .*

Proof: (i) \Rightarrow (iii) and (i) \Rightarrow (iv) are trivial.

(i) \Rightarrow (ii): On the one hand, from the definition of connected rotation-annihilation we know that B is a subuniverse. On the other hand, given a pair of elements $a, b \in B \setminus \{\bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}}\}$ such that $a < b$, we have $b \rightarrow a = \max\{c \in A : b \& c \leq a\} \notin F$; therefore $a/F \neq b/F$.

(ii) \Rightarrow (i): Assume (ii). If F and B do not give a decomposition, then there are $a \in B \setminus \{\bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}}\}$ and $b \in F$ such that $a \& b < a$. Then, $a \rightarrow a \& b \in F$, and hence $(a \& b)/F = x/F$; a contradiction.

(iii) \Rightarrow (ii): First let us check that B is a subuniverse. It is clear that it is closed under \neg . To show that it is also closed under $\&$, take a pair of elements $a, b \in B$ such that $a \leq b$. If $b \in A_-$, then $a \& b = \bar{0}^{\mathcal{A}}$. Suppose that $b \in A_+$ and $\neg a < b$. If $a \& b \in \neg F$, then $\bar{0}^{\mathcal{A}} = \neg(a \& b) \& (b \& a) = (\neg(a \& b) \& b) \& a = b \& a$; a contradiction. Now we show that $\mathcal{A}/F \cong \mathcal{B}$. If $a, b \in A_+ \setminus F$ with $a < b$, then $b \rightarrow a \notin F$, hence $a/F \neq b/F$. If $a, b \in A_- \setminus \neg F$, then $(\neg a)/F \neq (\neg b)/F$, thus $a/F \neq b/F$.

(iv) \Rightarrow (iii): Take $a \in F \setminus \{\bar{1}^{\mathcal{A}}\}$ and $b \in (B \cap A_+) \setminus \{\bar{1}^{\mathcal{A}}\}$ and we must show $a \& b = b$. Suppose that $a \& b < b$. Then, by the convexity and since $b^2 \leq a \& b \leq b$, we have $a \& b \in B$. Thus, $b \rightarrow a \& b \in B$, but $b \rightarrow a \& b \geq a$, a contradiction. \square

Unfortunately, the class of indecomposable IMTL-chains with respect to this decomposition is again far away from being described. We know that it contains all simple IMTL-chains as a proper subclass, but we do not have a general description for it. However, some particular cases of this decomposition will be studied in Chapter 6.

Chapter 5

Properties of varieties of MTL-algebras

Since the logics studied in this dissertation are algebraizable we have all the bridge theorems at our disposal, and hence we can study relevant logical properties of our axiomatic extensions of MTL by solving some algebraic problems of the corresponding subvarieties of MTL, and viceversa. In this chapter we focus on the logical and algebraic properties that we will consider in the following chapters.

5.1 Standard completeness. Methods and results.

In Chapter 3 left-continuous t-norms and their residua have been introduced in order to provide suitable semantics for fuzzy logics. Therefore, we are interested in completeness results with respect to the semantics given by the standard algebras.

Definition 5.1. *If a logic L is an algebraizable expansion of MTL in a language \mathcal{L}' , we say that L has the property of the (finite) strong standard completeness, (F)SSC for short, when for every (finite) set of formulae $T \subseteq \text{Fm}_{\mathcal{L}'}$ and every formula φ it holds that $T \vdash_L \varphi$ iff $T \models_{\mathcal{A}} \varphi$ for every standard L -algebra \mathcal{A} . We say that L has the property of standard completeness, SC for short, when the equivalence is true for $T = \emptyset$.*

Of course, the SSC implies the FSSC, and the FSSC implies the SC. On the scope of algebraizable logics, these properties have their equivalent algebraic property.

Theorem 5.2. *Let L be an axiomatic extension (or algebraizable axiomatic expansion) of MTL and let \mathbb{L} be its equivalent variety semantics. Then:*

1. \mathbb{L} has the SC if, and only if, $\mathbb{L} = \mathbf{V}(\text{Stand}_{\mathbb{L}})$, and
2. \mathbb{L} has the FSSC if, and only if, $\mathbb{L} = \mathbf{Q}(\text{Stand}_{\mathbb{L}})$,

where $\text{Stand}_{\mathbb{L}}$ is the class of all standard algebras in \mathbb{L} .

Proof: Both statements are proved in an analogous way. Let us prove the first one as a matter of example. Suppose first that \mathbb{L} has the SC and take an arbitrary equation in the language of \mathbb{L} , $\varphi \approx \psi \in Eq_{\mathcal{L}'}$. We have the following chain of equivalences: $\models_{\mathbb{L}} \varphi \approx \psi$ iff $\models_{\mathbb{L}} \varphi \leftrightarrow \psi \approx \bar{1}$ iff $\vdash_{\mathbb{L}} \varphi \leftrightarrow \psi$ iff $\models_{\text{Stand}_{\mathbb{L}}} \varphi \leftrightarrow \psi \approx \bar{1}$ iff $\models_{\text{Stand}_{\mathbb{L}}} \varphi \approx \psi$ iff $\models_{\mathbf{V}(\text{Stand}_{\mathbb{L}})} \varphi \approx \psi$. Therefore, $\mathbf{V}(\text{Stand}_{\mathbb{L}})$ and \mathbb{L} must be the same variety since they satisfy the same equations. Conversely, suppose $\varphi \in Fm_{\mathcal{L}'}$ is such that $\not\models_{\mathbb{L}} \varphi$. Then the equation $\varphi \approx \bar{1}$ is not valid in \mathbb{L} , so it also fails in $\text{Stand}_{\mathbb{L}}$. \square

There are also some equivalencies for the strong standard completeness.

Theorem 5.3. *Let \mathbb{L} be an axiomatic extension (or algebraizable axiomatic expansion) of MTL and let \mathbb{L} be its equivalent variety semantics. Then the following are equivalent:*

- (i) \mathbb{L} has the SSC.
- (ii) Every countable chain of \mathbb{L} belongs to $\mathbf{ISP}(\text{Stand}_{\mathbb{L}})$.
- (iii) Every countable subdirectly irreducible chain of \mathbb{L} is embeddable into a standard \mathbb{L} -chain.

Proof: (i) \Rightarrow (ii): Assume that (ii) is false, i.e. there is a countable \mathbb{L} -chain $\mathcal{A} \notin \mathbf{ISP}(\text{Stand}_{\mathbb{L}})$. Then, there exists a generalized quasiequation such that $\text{Stand}_{\mathbb{L}} \models \&_{i \in \kappa} p_i \approx q_i \Rightarrow p \approx q$ and $\mathcal{A} \not\models \&_{i \in \kappa} p_i \approx q_i \Rightarrow p \approx q$. There might be uncountably many variables occurring in the quasiequation but, since \mathcal{A} is countable, it can be arranged. Indeed, let $\{x_i : i \in \lambda\}$ be the variables of the quasiequation and consider a countable set of variables $\{y_i : i \in \omega\}$. Consider an enumeration $\{a_i : i \in \omega\}$ of the elements of \mathcal{A} and let e be any evaluation in \mathcal{A} such that it gives a countermodel of the generalized quasiequation. Now we define a mapping $\sigma : \{x_i : i \in \lambda\} \rightarrow \{y_i : i \in \omega\}$ by letting $\sigma(x_i) = y_j$ if $e(x_i) = a_j$, and we extend it to all formulae. It stills holds that $\text{Stand}_{\mathbb{L}} \models \&_{i \in \kappa} \sigma(p_i) \approx \sigma(q_i) \Rightarrow \sigma(p) \approx \sigma(q)$ and $\mathcal{A} \not\models \&_{i \in \kappa} \sigma(p_i) \approx \sigma(q_i) \Rightarrow \sigma(p) \approx \sigma(q)$. Therefore: $\{\sigma(p_i) \leftrightarrow \sigma(q_i) : i \in \kappa\} \models_{\text{Stand}_{\mathbb{L}}} \sigma(p) \leftrightarrow \sigma(q)$ and $\{\sigma(p_i) \leftrightarrow \sigma(q_i) : i \in \kappa\} \not\models_{\mathbb{L}} \sigma(p) \leftrightarrow \sigma(q)$, and hence the SSC does not hold.

(ii) \Rightarrow (i): Take arbitrary formulae $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}'}$ in the language of \mathbb{L} such that $\Gamma \models_{\text{Stand}_{\mathbb{L}}} \varphi$, i.e. $\{\psi \approx \bar{1} : \psi \in \Gamma\} \models_{\text{Stand}_{\mathbb{L}}} \varphi \approx \bar{1}$. Then, $\{\psi \approx \bar{1} : \psi \in \Gamma\} \models_{\mathbf{ISP}(\text{Stand}_{\mathbb{L}})} \varphi \approx \bar{1}$, and hence, by (ii), $\{\psi \approx \bar{1} : \psi \in \Gamma\} \models_{\{\text{countable } \mathbb{L}\text{-chains}\}} \varphi \approx \bar{1}$. Therefore, we have $\{\psi \approx \bar{1} : \psi \in \Gamma\} \models_{\{\mathbb{L}\text{-chains}\}} \varphi \approx \bar{1}$, and hence $\Gamma \vdash_{\mathbb{L}} \varphi$.

(i) \Rightarrow (iii): Let \mathcal{A} be a non-trivial countable subdirectly irreducible \mathbb{L} -chain. Let $F \subseteq A$ be its minimal non-trivial filter and take $x \in F \setminus \{\bar{1}^{\mathcal{A}}\}$. Consider a

set of pairwise different variables $\{v_a : a \in A\}$ and the following theory: $T = \{\lambda(v_{a_1}, \dots, v_{a_n}) \leftrightarrow v_{\lambda^A(a_1, \dots, a_n)} : \lambda \text{ } n\text{-ary connective, } a_1, \dots, a_n \in A\} \cup \{v_{\top}, \neg v_{\perp}\}$. We have $T \not\models_L v_x$ (just take \mathcal{A} with the evaluation $e(v_a) = a$ as a countermodel), therefore, by the SSC, there is a standard L-chain $[0, 1]_*$ and an evaluation e on $[0, 1]_*$ such that $e[T] = \{1\}$ and $e(v_x) < 1$. Consider the mapping $f : A \rightarrow [0, 1]$ defined as $f(a) = e(v_a)$. It is clear that f gives a homomorphism from \mathcal{A} to $[0, 1]_*$, and it is one-to-one because $f(x) \neq 1$ and x belongs to all the non-trivial filters.¹

(iii) \Rightarrow (i): Assume (iii) and take arbitrary formulae $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ in the language of L such that $\Gamma \not\models_L \varphi$. Then, there is a subdirectly irreducible L-chain \mathcal{A} and an evaluation e of the formulae on \mathcal{A} such that $e[\Gamma] \subseteq \{\bar{1}^{\mathcal{A}}\}$ and $e(\varphi) \neq \bar{1}^{\mathcal{A}}$. Let \mathcal{B} be the homomorphic image of $Fm_{\mathcal{L}}$ by e . Of course, e can be seen as an evaluation on \mathcal{B} , so \mathcal{B} (with e) is a countable countermodel for the derivation. Consider its representation as subdirect product of chains, $\mathcal{B} \hookrightarrow_{sp} \prod_{i \in I} \mathcal{B}_i$. Of course, every \mathcal{B}_i is countable, and there is some $j \in I$ such that $\pi_j(e(\varphi)) \neq \bar{1}^{\mathcal{B}_j}$. Now by (iii), \mathcal{B}_j is embeddable in some standard L-chain $[0, 1]_*$. Let f be the embedding. Then, $[0, 1]_*$ with the evaluation $f \circ \pi_j \circ e$ is a standard countermodel for the derivation, so we obtain $\Gamma \not\models_{Stand_L} \varphi$, as desired. \square

Standard completeness properties for axiomatic extensions of MTL has been a matter of intensive research. Table 5.1 shows the results obtained for the logics introduced in Chapter 3.

Table 5.1: Standard completeness properties of some axiomatic extensions of MTL.

Logic	SC	FSSC	SSC
MTL	Yes	Yes	Yes
IMTL	Yes	Yes	Yes
SMTL	Yes	Yes	Yes
IIMTL	Yes	Yes	No
BL	Yes	Yes	No
SBL	Yes	Yes	No
L	Yes	Yes	No
II	Yes	Yes	No
G	Yes	Yes	Yes
WNM	Yes	Yes	Yes
NM	Yes	Yes	Yes
CPC	No	No	No

It is obvious that Classical Propositional Calculus CPC does not enjoy any of the properties, because there are no standard algebras in its equivalent algebraic

¹This proof was found by Petr Cintula (private communication).

semantics (in fact, there are only two linearly ordered Boolean algebras: the trivial one and \mathcal{B}_2). The FSSC is proved in [86] for Łukasiewicz logic, in [83] for Product logic, in [47] for Gödel logic and in [30] for BL and SBL. In some cases (see for instance [83, 30]), rather than using the equivalences stated above, the result has been obtained by proving first that every chain of the equivalent variety semantics is partially embeddable into a standard algebra. For a long time, this condition was only known to be sufficient, but as we shall see, when the language is finite it is actually equivalent to the FSSC.

Proposition 5.4. *Let L be an axiomatic extension (or algebraizable axiomatic expansion in a finite language) of MTL. Then L has the FSSC if, and only if, every L -chain is partially embeddable into $Stand_L$.*

Proof: If L satisfies the FSSC then, by Theorem 5.2, its equivalent variety semantics \mathbb{L} is such that $\mathbb{L} = \mathbf{Q}(Stand_L)$. It follows from [42, Lemma 1.5] that every relative finitely subdirectly irreducible member of $\mathbf{Q}(Stand_L)$ belongs to $\mathbf{ISP}_U(Stand_L)$. Since $\mathbf{Q}(Stand_L)$ is a variety, relative finitely subdirectly irreducible members coincide with finitely subdirectly irreducible algebras in the absolute sense, hence with L -chains. Therefore, if L satisfies the FSSC, then every L -chain belongs to $\mathbf{ISP}_U(Stand_L)$ which is equivalent to partial embeddability by Proposition 2.8.

If every L -chain is partially embeddable into $Stand_L$, then by Proposition 2.8 every L -chain belongs to $\mathbf{ISP}_U(Stand_L)$. Now, since every L -algebra is representable as subdirect product of L -chains we have that

$$\mathbb{L} \subseteq \mathbf{IP}_{SD}(\mathbf{ISP}_U(Stand_L)) \subseteq \mathbf{Q}(Stand_L) \subseteq \mathbb{L}.$$

Therefore by Theorem 5.2, L has the FSSC. □

Notice that the implication from right to left of the last proposition is true even when the language is infinite.

Among all axiomatic extensions of BL, only G enjoys the SSC. As a matter of example, let $\Gamma = \{q \rightarrow p^n \mid n \in \mathbb{N}\}$ and $\varphi = (q \rightarrow q^2) \vee (p \rightarrow p^2) \vee (q \rightarrow p \& q)$, and consider the following semantical deduction: $\Gamma \models_{[0,1]_*} \varphi$. One can check that this deduction holds for every continuous t-norm $*$, but $\Gamma \not\models_{L_*} \varphi$, since $\{\psi \approx \bar{1} : \psi \in \Gamma_0\} \not\models_{[0,1]_*} \varphi \approx \bar{1}$, for every finite $\Gamma_0 \subseteq \Gamma$ when $*$ \neq min.

The SSC has been proved for several axiomatic extensions of MTL, as we can see in Table 5.1. But instead of using the equivalencies of Theorem 5.3, the usual strategy has consisted on proving a stronger property, namely showing in a constructive way that every countable chain is embeddable into a standard chain of the same variety. This kind of construction was first introduced by Jenei and Montagna in [100] in order to prove the SSC for MTL. We will sketch it now, because it will be used in the following chapters.

Completion of countable MTL-chains:

Take an arbitrary countable MTL-chain \mathcal{A} . A standard MTL-chain $[0, 1]_*$ and an embedding $h : \mathcal{A} \hookrightarrow [0, 1]_*$ are built by following the next steps:

- Consider the set $B := \{\langle \bar{0}^A, 1 \rangle\} \cup \{\langle a, q \rangle : a \in A \setminus \{\bar{0}^A\}, q \in \mathbb{Q} \cap (0, 1]\}$.
- Consider the lexicographical order \preceq on B .
- Define the following monoidal operation on B :

$$\langle a, q \rangle \circ \langle b, r \rangle := \begin{cases} \min_{\preceq} \{\langle a, q \rangle, \langle b, r \rangle\} & \text{if } a \&^A b = \min\{a, b\} \\ \langle a \&^A b, 1 \rangle & \text{otherwise.} \end{cases}$$

- The ordered monoid $\langle A, \&^A, \bar{1}^A, \preceq \rangle$ is embeddable into $\langle B, \circ, \langle \bar{1}^A, 1 \rangle, \preceq \rangle$ by mapping every $a \in A$ to $\langle a, 1 \rangle$.
- $\mathcal{B} = \langle B, \circ, \langle \bar{1}^A, 1 \rangle, \preceq \rangle$ is a densely ordered countable monoid with maximum and minimum, so it is isomorphic to a monoid $\mathcal{B}' = \langle \mathbb{Q} \cap [0, 1], \circ', 1, \preceq' \rangle$. Obviously, $\langle A, \&^A, \bar{1}^A, \preceq \rangle$ is also embeddable into \mathcal{B}' . Let h be such embedding. Moreover, restricted to $h[A]$, the residuum of \circ' exists, call it \Rightarrow , and $h(a) \Rightarrow h(b) = h(a \rightarrow^A b)$.
- \mathcal{B}' is completed to $[0, 1]$ by defining:

$$\forall \alpha, \beta \in [0, 1] \quad \alpha * \beta := \sup\{x \circ' y : x \leq \alpha, y \leq \beta, x, y \in \mathbb{Q} \cap [0, 1]\}.$$

- $*$ is a left-continuous t-norm, so it defines a standard MTL-algebra $[0, 1]_*$, and h is the desired embedding. $[0, 1]_*$ is called the *completion* of \mathcal{A} .

Therefore, we obtain the following sufficient condition for the SSC:

Proposition 5.5. *Let L be an axiomatic extension (or algebraizable axiomatic expansion) of MTL. If for every countable L -chain, its completion given by Jenei and Montagna construction is an L -chain, then L enjoys the SSC.*

The SSC for MTL, SMTL, G and WNM can be proved by applying the previous proposition. Nevertheless, it does not work for IMTL, NM and IIMTL. In [49] the authors prove that the completion of Jenei and Montagna does not preserve neither the involution nor the cancellation law in general. But the SSC for IMTL can still be proved by modifying the construction. We sketch it again.

Completion of countable involutive MTL-chains:

Let \mathcal{A} be a countable IMTL-chain. A standard IMTL-chain $[0, 1]_*$ and an embedding $h : \mathcal{A} \hookrightarrow [0, 1]_*$ are built by following the next steps:

- For every $a \in A$, $\text{suc}(a)$ is defined the successor of a in the order of \mathcal{A} if it exists, or $\text{suc}(a) = a$ otherwise.
- $B := \{\langle a, 1 \rangle : a \in A\} \cup \{\langle a, q \rangle : \exists a' \in A \text{ such that } a \neq a' \text{ and } \text{suc}(a') = a, q \in \mathbb{Q} \cap (0, 1)\}$.
- Consider the lexicographical order \preceq on B .

- As before, we define the following monoidal operation on B :

$$\langle a, q \rangle \circ \langle b, r \rangle := \begin{cases} \min_{\prec} \{ \langle a, q \rangle, \langle b, r \rangle \} & \text{if } a \&^A b = \min\{a, b\} \\ \langle a \&^A b, 1 \rangle & \text{otherwise.} \end{cases}$$

but now the operation is modified in the following way:

$$\langle a, q \rangle \otimes \langle b, r \rangle := \begin{cases} \langle \bar{0}^A, 1 \rangle & \text{if } a = \text{suc}(\neg b), q + r \leq 1 \\ \langle a, q \rangle \circ \langle b, r \rangle & \text{otherwise.} \end{cases}$$

- The ordered monoid $\langle A, \&^A, \bar{1}^A, \leq \rangle$ is embeddable into $\langle B, \otimes, \langle \bar{1}^A, 1 \rangle, \preceq \rangle$ by mapping every $a \in A$ to $\langle a, 1 \rangle$.
- $\mathcal{B} = \langle B, \otimes, \langle \bar{1}^A, 1 \rangle, \preceq \rangle$ is a densely ordered countable monoid with maximum and minimum, so it is isomorphic to a monoid $\mathcal{B}' = \langle \mathbb{Q} \cap [0, 1], \circ', \preceq' \rangle$. Obviously, $\langle A, \&^A, \bar{1}^A, \leq \rangle$ is also embeddable into \mathcal{B}' . Let h be such embedding. As before, restricted to $h[A]$, the residuum of \circ' exists, call it \Rightarrow , and $h(a) \Rightarrow h(b) = h(a \rightarrow^A b)$. Moreover, for every $q \in \mathbb{Q} \cap [0, 1]$, the residuum of q and 0, $q \Rightarrow 0$, exists, and the operation $n(q) = q \Rightarrow 0$ is an involutive negation on \mathcal{B}' .
- \mathcal{B}' is completed to $[0, 1]$ by defining:

$$\forall \alpha, \beta \in [0, 1] \quad \alpha * \beta := \sup\{x \circ' y : x \leq \alpha, y \leq \beta, x, y \in \mathbb{Q} \cap [0, 1]\}.$$

- $*$ is a left-continuous t-norm with an involutive negation, so it defines a standard IMTL-algebra $[0, 1]_*$, and h is the desired embedding.

The SSC fails for IIMTL, as we will see in Chapter 7, but this logic still enjoys the FSSC as it was proved in [88].

Sometimes standard completeness properties can be refined to some subclass of standard algebras; sometimes even it is enough to consider only one standard algebra. When the standard completeness can be proved with respect to a particular standard algebra which is the intended semantics for the logic, we call it *canonical* standard completeness. Notice that in theorems 5.2, 5.3 and 5.4, the equivalencies remain true when restricted to some subclass of standard algebras. The canonical standard completeness is a matter of special interest when one considers the logic of the variety generated by the algebra defined by one particular t-norm, because then this t-norm gives the intended semantics for the logic.

Definition 5.6. Let $*$ be a left-continuous t-norm. L_* will denote the axiomatic extension of MTL whose equivalent algebraic semantics is $\mathbf{V}([0, 1]_*)$, the variety generated by $[0, 1]_*$.

It is clear, by definition, that for every left-continuous t-norm $*$, the logic L_* enjoys the SC restricted to $[0, 1]_*$, i.e. the canonical SC. Sometimes the standard

completeness is stronger. As we have already mentioned, for instance, we have canonical FSSC for \mathbf{L} and \mathbf{IL} , and canonical SSC for \mathbf{G} and \mathbf{NM} .

Actually, given any continuous t-norm $*$, it follows from Corollary 4.53 that all chains in $\mathbf{V}([0, 1]_*)$ are partially embeddable into $[0, 1]_*$. Therefore, after Proposition 5.4 we obtain the following direct corollary.

Corollary 5.7. *For every continuous t-norm $*$, the logic \mathbf{L}_* has the canonical FSSC.*

Nevertheless, this result cannot be improved to SSC. Among all the continuous t-norm based logics, only \mathbf{G} enjoys the SSC.

Proposition 5.8. *For every axiomatic extension \mathbf{L} of \mathbf{BL} , \mathbf{L} has the SSC if, and only if, $\mathbf{L} = \mathbf{G}$.*

Proof: Let $\Gamma = \{q \rightarrow p^n \mid n \in \mathbb{N}\}$ and $\varphi = (q \rightarrow q^2) \vee (p \rightarrow p^2) \vee (q \rightarrow p \& q)$, and consider the following semantical deduction: $\Gamma \models_{[0,1]_*} \varphi$. One can check that this deduction holds for every continuous t-norm $*$, but $\Gamma \not\models_{\mathbf{L}_*} \varphi$, since $\Gamma_0 \not\models_{[0,1]_*} \varphi$, for every finite $\Gamma_0 \subseteq \Gamma$ when $*$ \neq min. \square

To end up this section we show that if an axiomatic extension of \mathbf{MTL} does not enjoy the SC, the FSSC or the SSC, then any of its conservative expansions neither does.

Proposition 5.9. *Suppose that \mathbf{L}' is a conservative expansion of \mathbf{L} . Then:*

- *If \mathbf{L}' enjoys the SC, then \mathbf{L} enjoys the SC.*
- *If \mathbf{L}' enjoys the FSSC, then \mathbf{L} enjoys the FSSC.*
- *If \mathbf{L}' enjoys the SSC, then \mathbf{L} enjoys the SSC.*

Proof: All the implications are proved in a similar way. Let us prove as an example the first one. Suppose that \mathbf{L} does not enjoy the SC. Then, there is a formula $\varphi \in \mathbf{Fm}_{\mathcal{L}}$ such that $\not\models_{\mathbf{L}} \varphi$ and $\models_{\mathcal{C}} \varphi$ for every standard \mathbf{L} -chain \mathcal{C} . Let \mathcal{A} be a standard \mathbf{L}' -chain. Then, its \mathcal{L} -reduct is a model of φ , thus $\models_{\mathcal{A}} \varphi$ and, since \mathbf{L}' is a conservative expansion of \mathbf{L} , we also have $\not\models_{\mathbf{L}'} \varphi$. Therefore, \mathbf{L}' does not enjoy the SC. \square

5.2 Other algebraic and logical properties

Regarding to other algebraic properties that will be used in the dissertation, in the previous chapter we have seen that \mathbf{MTL} and its subvarieties enjoy the CEP. Moreover, it is easy to prove that all these varieties are also arithmetic. Actually, one can check that \mathbf{RL} has the following 2/3-minority term: $m(x, y, z) = ((x \rightarrow y) \rightarrow z) \wedge ((z \rightarrow y) \rightarrow x) \wedge (x \vee z)$ (cf. [95]).

Other meaningful algebraic and logical properties for our logics and varieties are not so easy to study and must be discussed in a more detailed basis. We

will focus on the local finiteness, the FEP, the FMP and decidability. They have been studied for several axiomatic extensions of MTL in the literature, but the results are quite spread. We summarize all of them here.

As regards to local finiteness, it is trivial for CPC and G, proved for NM in [71] and generalized to WNM in Chapter 9 of this dissertation. It is easy to refute it for the rest of the logics so far considered. On the one hand, take any $a \in (0, 1)$ in the standard product algebra. The subalgebra of $[0, 1]_{\Pi}$ generated by a is clearly infinite, therefore the following logics are not locally finite: Π , SBL, BL, IIMTL, SMTL and MTL. On the other hand, let a be a positive element in Chang's MV-algebra \mathcal{C} such that $c \neq \bar{1}^{\mathcal{C}}$. Then, the subalgebra of \mathcal{C} generated by a is again infinite, therefore local finiteness fails also for L and IMTL.

Since the FEP follows from the local finiteness, we do not need to discuss it for WNM and its axiomatic extensions. It has been proved for L (see [17]), BL and SBL (see [2] and [116]). However, since there are no finite Π -chains and IIMTL-chains (except for the trivial one and \mathcal{B}_2), the FMP (hence also the FEP) fails for IIMTL and Π . The FEP also holds for MTL, IMTL and SMTL. Actually, it was first proved for the Monoidal Logic by Blok and Van Alten in [22], and then Ono² used the same construction for MTL, IMTL and SMTL. The proof has been improved in [28].

Almost all of the so far considered logics are decidable because they enjoy the FMP. We have seen that for only two of them the FMP fails: Product logic and IIMTL. However, Π is still decidable due to its connection to lattice ordered Abelian groups (see [79]), and the decidability of IIMTL has been proved very recently in [90]. All the results are gathered in Table 5.2.

Table 5.2: Some algebraic and logical properties.

Logic	LF	FEP	FMP	Decidable
MTL	No	Yes	Yes	Yes
IMTL	No	Yes	Yes	Yes
SMTL	No	Yes	Yes	Yes
IIMTL	No	No	No	Yes
BL	No	Yes	Yes	Yes
SBL	No	Yes	Yes	Yes
L	No	Yes	Yes	Yes
Π	No	No	No	Yes
G	Yes	Yes	Yes	Yes
WNM	Yes	Yes	Yes	Yes
NM	Yes	Yes	Yes	Yes
CPC	Yes	Yes	Yes	Yes

²His proof has never been published. We know it by private communication.

Chapter 6

Perfect, local and bipartite MTL-algebras

In Chapter 4 we have presented Jenei's methods to construct new families of IMTL-algebras. This led us to some considerations on a possible representation theorem for IMTL-chains in terms of decomposition as connected rotation-annihilation of some filter and a convex subalgebra, but we do not have a description of indecomposable chains yet. In this chapter we will generalize the connected rotation-annihilation to MTL-chains and we will show that in some cases the decomposition can be well described, namely those where the used filter is the radical and the convex subalgebra is \mathcal{B}_2 or the so-called *drastic product* algebras. These kinds of chains will be studied in this chapter by generalizing the notions of perfect, local and bipartite algebra that have been already used for MV-algebras and BL-algebras (see [5, 13, 45, 46, 135]).

6.1 Perfect and bipartite MTL-algebras

We start with the notion of order of an element, which will allow us to define the class of perfect algebras.

Definition 6.1. *Let \mathcal{A} be an MTL-algebra. We define the order of $a \in A$ as:*

$$\text{ord}(a) = \begin{cases} \min\{n : a^n = \bar{0}^{\mathcal{A}}\} & \text{if it exists,} \\ \infty & \text{otherwise.} \end{cases}$$

Definition 6.2. *An MTL-algebra \mathcal{A} is perfect if, and only if, for every $a \in A$, $\text{ord}(a) < \infty$ iff $\text{ord}(-a) = \infty$.*

Some easy examples of perfect MTL-algebras are \mathcal{B}_2 , Chang's algebra \mathcal{C} , WNM-chains without negation fixpoint and all SMTL-chains. Notice that perfect algebras cannot have negation fixpoint.

Proposition 6.3. *Let $F \subseteq A$ be a filter of an MTL-algebra \mathcal{A} . Then the subuniverse of A generated by F is $F \cup \bar{F}$.*

Definition 6.4. *An MTL-algebra \mathcal{A} is bipartite if, and only if, there is a maximal filter $F \subseteq A$ such that $A = F \cup \bar{F}$. In this case we say that \mathcal{A} is bipartite by F .*

Definition 6.5. *Let \mathcal{A} be an MTL-algebra. $\mathcal{A} \in \mathbb{BP}_0$ if, and only if, for every $F \in \text{Max}(\mathcal{A})$, $A = F \cup \bar{F}$, i.e. \mathcal{A} is bipartite by all maximal filters. We also define the corresponding class of IMTL-algebras as $\mathbb{IBP}_0 := \mathbb{BP}_0 \cap \text{IMTL}$.*

As perfect algebras, bipartite algebras do not have negation fixpoint. Notice that all the examples of perfect algebras mentioned before are also in \mathbb{BP}_0 . \mathcal{B}_4 is an example of an algebra in \mathbb{IBP}_0 which is not perfect. Besides, not all bipartite algebras are in \mathbb{BP}_0 ; for instance, $L_3 \times \mathcal{B}_2$ and $G_3 \times \mathcal{B}_2$ are bipartite algebras (involutive and non-involutive, respectively) that are not in \mathbb{BP}_0 .

However, for MTL-chains, perfect and bipartite algebras and algebras from \mathbb{BP}_0 turn out to be the same:

Theorem 6.6. *Let \mathcal{A} be an MTL-chain. The following are equivalent:*

- (1) $A = \text{Rad}(\mathcal{A}) \cup \overline{\text{Rad}(\mathcal{A})}$.
- (2) \mathcal{A} is bipartite.
- (3) $\mathcal{A} \in \mathbb{BP}_0$.
- (4) $\text{Rad}(\mathcal{A}) = A_+$ and \mathcal{A} has no fixpoint.
- (5) \mathcal{A} is perfect.
- (6) $\mathcal{A} \models Bp(x) \approx \bar{1}$.
- (7) $\mathcal{A}/\text{Rad}(\mathcal{A}) \cong \mathcal{B}_2$.

where $Bp(x) = (\neg(\neg x)^2)^2 \leftrightarrow \neg(\neg x^2)^2$.

Proof: (1) \Rightarrow (2), (2) \Rightarrow (3), (3) \Rightarrow (4) and (4) \Rightarrow (5) are straightforward.

(5) \Rightarrow (6): If the chain is perfect, then one can check that for every $a \in A_+$, $(\neg(\neg a)^2)^2 = \neg(\neg a^2)^2 = \bar{1}^{\mathcal{A}}$ and for every $a \in A_-$, $(\neg(\neg a)^2)^2 = \neg(\neg a^2)^2 = \bar{0}^{\mathcal{A}}$.

(6) \Rightarrow (7): Suppose that \mathcal{A} satisfies the equation. Notice that in this case the set of positive elements is a proper filter. Indeed, if $a \in A_+$, then $\neg a \in A_-$, so $(\neg a)^2 = \bar{0}^{\mathcal{A}}$. Therefore $(\neg(\neg a)^2)^2 = \bar{1}^{\mathcal{A}} = \neg(\neg a^2)^2$ and this implies $a^2 \in A_+$. Now, take $a, b \in A_+$ such that $a \leq b$. Then $a^2 \leq a \& b$ and $a^2 \in A_+$, so $a \& b \in A_+$. Thus $A_+ = \text{Rad}(\mathcal{A})$. Consider the algebra $\mathcal{A}/\text{Rad}(\mathcal{A})$ and take $a \in A$. If a is positive, then $a \rightarrow \bar{1}^{\mathcal{A}} = \bar{1}^{\mathcal{A}} \in \text{Rad}(\mathcal{A})$ and $\bar{1}^{\mathcal{A}} \rightarrow a = a \in \text{Rad}(\mathcal{A})$, so $a/\text{Rad}(\mathcal{A}) = \bar{1}^{\mathcal{A}}/\text{Rad}(\mathcal{A})$. If a is negative, then $a \rightarrow \bar{0}^{\mathcal{A}} = \neg a \in \text{Rad}(\mathcal{A})$ and $\bar{0}^{\mathcal{A}} \rightarrow a = \bar{1}^{\mathcal{A}} \in \text{Rad}(\mathcal{A})$, so $a/\text{Rad}(\mathcal{A}) = \bar{0}^{\mathcal{A}}/\text{Rad}(\mathcal{A})$. Therefore $\mathcal{A}/\text{Rad}(\mathcal{A}) \cong \mathcal{B}_2$.

(7) \Rightarrow (1): Suppose that the quotient by the radical is the two element Boolean algebra. Take an arbitrary $a \in A$ and suppose $a \notin \text{Rad}(\mathcal{A})$. Then $a/\text{Rad}(\mathcal{A}) \neq \bar{1}^A/\text{Rad}(\mathcal{A})$, so $a/\text{Rad}(\mathcal{A}) = \bar{0}^A/\text{Rad}(\mathcal{A})$ and hence $\neg a/\text{Rad}(\mathcal{A}) = \bar{1}^A/\text{Rad}(\mathcal{A})$, i.e. $\neg a \in \text{Rad}(\mathcal{A})$. \square

Theorem 6.7. *Let \mathcal{A} be an MTL-algebra. Then:*

\mathcal{A} is perfect if, and only if, $A = \text{Rad}(\mathcal{A}) \cup \overline{\text{Rad}(\mathcal{A})}$.

Proof: Suppose that \mathcal{A} is perfect. By Corollary 6.25 we know that $\text{Rad}(\mathcal{A}) = \{a \in A : \text{ord}(a) = \infty\}$ and then the result follows immediately. Conversely, if $A = \text{Rad}(\mathcal{A}) \cup \overline{\text{Rad}(\mathcal{A})}$ then every $a \in \text{Rad}(\mathcal{A})$ has infinite order and every $a \in \overline{\text{Rad}(\mathcal{A})}$ has finite order, hence the algebra is perfect. \square

Corollary 6.8. *Every perfect algebra is bipartite.*

Proof: If the algebra is perfect, then it is local, so the radical is the only maximal filter and the result is obvious. \square

Another easy consequence is the following proposition about perfect subalgebras:

Corollary 6.9. *Given an MTL-algebra \mathcal{A} , $\text{Rad}(\mathcal{A}) \cup \overline{\text{Rad}(\mathcal{A})}$ is a perfect subalgebra and contains all perfect subalgebras.*

Theorem 6.10. *Let \mathcal{A} be an MTL-algebra. Then the following are equivalent:*

- (1) \mathcal{A} is perfect.
- (2) $\mathcal{A}/\text{Rad}(\mathcal{A}) \cong \mathcal{B}_2$.

Proposition 6.11. *Let \mathcal{A} be an MTL-algebra and let $M \subseteq A$ be a prime filter. Then the following are equivalent:*

- (1) $A_+ \subseteq M$ and \mathcal{A} has no fixpoint.
- (2) M is maximal and $A = M \cup \overline{M}$.
- (3) $\mathcal{A}/M \cong \mathcal{B}_2$.

Proof: (1) \Rightarrow (2): If $a \in A$, then by Proposition 4.12, $a \vee \neg a \in A_+ \subseteq M$, but since M is prime, $a \in M$ or $\neg a \in M$.

(2) \Rightarrow (3): On one hand, M is prime, so \mathcal{A}/M is a chain. On the other hand, for every $a \in A$, $a/M \vee \neg(a/M) = (a \vee \neg a)/M = \bar{1}^A/M$, hence \mathcal{A}/M is Boolean, so it must be the two element Boolean algebra.

(3) \Rightarrow (1): Take any $a \vee \neg a \in A_+$. $(a \vee \neg a)/M = a/M \vee \neg(a/M) = \bar{1}^A/M$, so $a \vee \neg a \in M$. \square

Lemma 6.12. *Let \mathcal{A} be an MTL-algebra and let $F \subseteq A$ be a proper filter. Then: $\mathcal{A}/F \in \mathbb{BA}$ if, and only if, $\{a \vee \neg a : a \in A\} \subseteq F$.*

Proof: Suppose that the quotient is a Boolean algebra and take $a \in A_+$. Then $a/F \vee \neg(a/F) = (a \vee \neg a)/F = \bar{1}^A/F$. Thus: $a = a \vee \neg a \in F$. Conversely, it is straightforward to check that \mathcal{A}/F satisfies the law of the excluded middle. \square

Theorem 6.13. *For every MTL-algebra \mathcal{A} the following are equivalent:*

- (1) $\mathcal{A} \in \mathbb{BP}_0$.
- (2) $\mathcal{A}/\text{Rad}(\mathcal{A}) \in \mathbb{BA}$.
- (3) $\text{Rad}(\mathcal{A}) = A_+$ and \mathcal{A} has no fixpoint.

Proof: (1) \Leftrightarrow (2): For every maximal filter M , $A = M \cup \overline{M}$ iff (by Theorem 6.11) A_+ is contained in every maximal filter iff $A_+ \subseteq \text{Rad}(\mathcal{A})$. By Lemma 6.12, this is equivalent to $\mathcal{A}/\text{Rad}(\mathcal{A}) \in \mathbb{BA}$.

(2) \Rightarrow (3): By Lemma 6.12, we obtain $A_+ \subseteq \text{Rad}(\mathcal{A})$ and the other inclusion is always true.

(3) \Rightarrow (2): Also by Lemma 6.12. \square

From (2) of the last theorem and Theorem 6.10 we obviously obtain the following result:

Corollary 6.14. *Every perfect MTL-algebra is in \mathbb{BP}_0 .*

Theorem 6.15. *\mathbb{BP}_0 is a variety. One equational base is obtained by adding the next set of equations to the usual axiomatization for MTL:*

$$\{(\neg x \wedge \neg \neg x) \rightarrow (x \vee \neg x)^n \approx \bar{1} : n \geq 1\}$$

Proof: Let \mathcal{A} be an MTL-algebra. $\mathcal{A} \in \mathbb{BP}_0$ iff $A_+ \subseteq \text{Rad}(\mathcal{A})$ and there is no fixpoint iff for every $a \vee \neg a \in A_+$ and every $n \geq 1$, $(a \vee \neg a)^n \geq \neg(a \vee \neg a) = \neg a \wedge \neg \neg a$. \square

Corollary 6.16. *\mathbb{BP}_0 is the variety generated by all perfect MTL-algebras.*

Proof: Let \mathbb{K} be the variety generated by all perfect MTL-algebras. By Corollary 6.14, $\mathbb{K} \subseteq \mathbb{BP}_0$. The other inclusion follows from the subdirect representation theorem and Theorem 6.6. \square

Corollary 6.17. *There is a simpler axiomatization for \mathbb{BP}_0 obtained by adding to the axioms of MTL only the equation $Bp(x) \approx \bar{1}$.*

Proof: Let \mathbb{K} be the variety of MTL-algebras satisfying this equation. We will prove $\mathbb{K} = \mathbb{BP}_0$. If $\mathcal{A} \in \mathbb{K}$, then by the subdirect representation theorem \mathcal{A} is representable as a subdirect product of chains satisfying the equation. By Theorem 6.6, these chains are in \mathbb{BP}_0 , so $\mathcal{A} \in \mathbb{BP}_0$. Conversely, take $\mathcal{A} \in \mathbb{BP}_0$. Then \mathcal{A} is isomorphic to a subdirect product of MTL-chains in \mathbb{BP}_0 , so it satisfies the equation. \square

We can prove the following Glivenko-style theorem¹ for the logic \mathbb{BP}_0 associated to the variety \mathbb{BP}_0 :

Theorem 6.18. *Let \vdash_{CPC} denote the relation of derivability in the classical propositional calculus. Then, for every $\varphi \in \text{Fm}_{\mathcal{L}}$, $\vdash_{\text{CPC}} \varphi$ if, and only if, $\vdash_{\mathbb{BP}_0} (\neg(\neg\varphi)^2)^2$.*

Proof: Suppose that $\vdash_{\text{CPC}} \varphi$. It suffices to prove that for each chain $\mathcal{A} \in \mathbb{BP}_0$, $\mathcal{A} \models (\neg(\neg\varphi)^2)^2 \approx \bar{1}$. Let \mathcal{A} be such a chain and $v : \text{Fm}_{\mathcal{L}} \rightarrow \mathcal{A}$ an evaluation. We know that $\mathcal{A}/\text{Rad}(\mathcal{A}) \cong \mathcal{B}_2$, so $v(\varphi)/\text{Rad}(\mathcal{A}) = \bar{1}^{\mathcal{A}}/\text{Rad}(\mathcal{A})$, i.e. $v(\varphi) \in A_+$, hence $(\neg(\neg v(\varphi))^2)^2 = \bar{1}^{\mathcal{A}}$. Conversely, if $\vdash_{\mathbb{BP}_0} (\neg(\neg\varphi)^2)^2$, then $\mathcal{B}_2 \models (\neg(\neg\varphi)^2)^2 \approx \bar{1}$, i.e. $\mathcal{B}_2 \models \varphi \approx \bar{1}$, hence $\vdash_{\text{CPC}} \varphi$. \square

Concerning the structure of the class of bipartite MTL-algebras, we obtain the following results:

Proposition 6.19. *The class of bipartite MTL-algebras is closed under subalgebras.*

Theorem 6.20. *Let $\{\mathcal{A}_i : i \in I\}$ be a set of MTL-algebras and take their direct product \mathcal{A} . If there is some $j \in I$ such that \mathcal{A}_j is bipartite, then \mathcal{A} is bipartite.*

Proof: Using the same reasoning as in Theorem 4.5 of [45]. \square

Corollary 6.21. *The class of bipartite MTL-algebras is closed under direct products.*

Corollary 6.22. *The variety generated by all bipartite MTL-algebras is MTL.*

Proof: Let \mathcal{A} be an arbitrary MTL-algebra. Consider $\mathcal{A} \times \mathcal{B}_2$, that is a bipartite MTL-algebra since \mathcal{B}_2 is bipartite. Thus, taking the projection over the first component, we obtain \mathcal{A} as a homomorphic image of a bipartite algebra. Therefore, every MTL-algebra is in the variety generated by all bipartite algebras. \square

6.2 Local MTL-algebras

The definition of local algebras is also done in terms of the order of the elements.

Definition 6.23. *An MTL-algebra \mathcal{A} is local if, and only if, for every $a \in A$ $\text{ord}(a) < \infty$ or $\text{ord}(\neg a) < \infty$.*

It is clear that all the chains are local algebras. Perfect algebras are local as well. In fact, we have this characterization:

Proposition 6.24. *An MTL-algebra is local if, and only if, it has a unique maximal filter.*

¹For a general study of this type of theorems in the framework of natural expansions of BCK logic see [34]. See also some results in [33].

Proof: Suppose that M is the unique maximal filter of \mathcal{A} . If there is $a \in A$ such that $\text{ord}(a) = \text{ord}(\neg a) = \infty$ then $a, \neg a \in M$ and this is a contradiction since M is proper. Conversely, suppose that \mathcal{A} is local and let M be a maximal filter. Then it is easy to prove that $M = \{a \in A : \text{ord}(a) = \infty\}$. Clearly M is contained in this set. If $a \notin M$ and $\text{ord}(a) = \infty$, then $\exists n$ such that $\neg a^n \in M$, so $\text{ord}(\neg a^n) = \infty$. Hence $\text{ord}(a^n) < \infty$, so $\text{ord}(a) < \infty$, a contradiction. \square

Corollary 6.25. *Let \mathcal{A} be an MTL-algebra.*

\mathcal{A} is local if, and only if, $\text{Rad}(\mathcal{A}) = \{a \in A : \text{ord}(a) = \infty\}$.

In order to state a classification theorem of local algebras, we define two new classes of MTL-algebras.

Definition 6.26. *An MTL-algebra \mathcal{A} is locally finite² if, and only if, for every $a \in A \setminus \{\bar{1}^{\mathcal{A}}\}$ $\text{ord}(a) < \infty$. \mathcal{A} is peculiar iff is local and $\exists a, b \in A \setminus \{\bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}}\}$ such that $\text{ord}(a) = \infty$, $\text{ord}(b) < \infty$ and $\text{ord}(\neg b) < \infty$.*

Theorem 6.27. *Let \mathcal{A} be a local MTL-algebra such that $\mathcal{A} \not\cong \mathcal{B}_2$. Then \mathcal{A} satisfies one, and only one, of the following:*

- \mathcal{A} is perfect.
- \mathcal{A} is locally finite.
- \mathcal{A} is peculiar.

We know that perfect algebras cannot have negation fixpoint but this is not the case for the other types of local algebras. For instance, on the one hand, $[0, 1]_{\mathbf{L}}$ is a locally finite MTL-algebra and, on the other hand, $[0, 1]_{\mathbf{NM}}$ and in general all WNM-chains with negation fixpoint are peculiar.

6.3 Perfect IMTL-algebras and disconnected rotations of prelinear semihoops

In the involutive case, the notion of perfect algebra turns out to be an exact description of the IMTL-chains obtained by means of the disconnected rotation method presented in Chapter 4. Recall that the disconnected rotation of a prelinear semihoop \mathcal{B} is denoted as \mathcal{B}^* .

Theorem 6.28. *Let \mathcal{A} be an IMTL-algebra. Then the following are equivalent:*

- (1) \mathcal{A} is perfect.
- (2) $\mathcal{A}/\text{Rad}(\mathcal{A}) \cong \mathcal{B}_2$.
- (3) \mathcal{A} is isomorphic to the disconnected rotation of a prelinear semihoop.

²We follow here the nomenclature introduced by Chang for MV-algebras. Do not confuse this with the general notion of locally finite algebra in Universal Algebra defined in Chapter 2.

Proof: (1) \Rightarrow (2): If the algebra is perfect, then the radical is perfect and maximal, hence $\mathcal{A}/\text{Rad}(\mathcal{A})$ is simple and perfect, so it must be isomorphic to \mathcal{B}_2 .

(2) \Rightarrow (3): For every $a \in A$, $(a/\text{Rad}(\mathcal{A}) = \bar{1}^A/\text{Rad}(\mathcal{A}) \Rightarrow a \in \text{Rad}(\mathcal{A}))$ and $(a/\text{Rad}(\mathcal{A}) = \bar{0}^A/\text{Rad}(\mathcal{A}) \Rightarrow a \in \neg\text{Rad}(\mathcal{A}))$. So $A = \text{Rad}(\mathcal{A}) \cup \neg\text{Rad}(\mathcal{A})$. Then, considering the prelinear semihoop \mathcal{B} given by $\text{Rad}(\mathcal{A})$, we obtain that $\mathcal{A} \cong \mathcal{B}^*$.

(3) \Rightarrow (1): If $\mathcal{A} \cong \mathcal{B}^*$ for some prelinear semihoop \mathcal{B} , then it is obvious that all positive elements have infinite order and all negative elements have finite order. \square

Furthermore, we can prove that perfect MV-algebras are exactly the disconnected rotations of cancellative hoops.

Theorem 6.29. *Let \mathcal{A} be an IMTL-algebra. The following are equivalent:*

(1) \mathcal{A} is a perfect MV-algebra.

(2) \mathcal{A} is isomorphic to the disconnected rotation of a cancellative hoop.

Proof: (1) \Rightarrow (2): It is clear that $\text{Rad}(\mathcal{A})$ is a Wajsberg hoop. We only need to show that it has no minimum element and then it will be a cancellative hoop. Suppose that a is the minimum of $\text{Rad}(\mathcal{A})$ and take any $x < a$ different from $\bar{0}^A$. Then $a \wedge x = x$, but $a \& (a \rightarrow x) = a \& \neg(a \& \neg x) = \neg(a \rightarrow a \& \neg x) = \neg(a \rightarrow a) = \bar{0}^A$, a contradiction with \mathcal{A} being a MV-algebra.

(2) \Rightarrow (1): The rotation of a cancellative hoop is always an MV-algebra as it is proved in Lemma 3.13 of [52] and by the last theorem it is perfect. \square

6.4 Correspondence between varieties of prelinear semihoops and varieties of IMTL-algebras

In this section we will prove that the lattice of subvarieties of \mathbb{IBP}_0 is really big and complex. Indeed we show that it is isomorphic to the lattice of subvarieties of prelinear semihoops.

We will need the next notation: given a class of prelinear semihoops \mathbb{K} , we define $\mathbb{K}^* := \{\mathcal{A}^* : \mathcal{A} \in \mathbb{K}\} \subseteq \text{IMTL}$.

Lemma 6.30. *If \mathbb{K} is a class of prelinear semihoops, then $\mathbf{H}(\mathbb{K}^*) = \mathbf{H}(\mathbb{K})^*$.*

Proof: Take $\mathcal{A} \in \mathbb{K}$ and $\mathcal{B} \in \mathbf{H}(\mathcal{A})$ and consider \mathcal{B}^* . We know that \mathcal{B} is the image of some homomorphism $h : \mathcal{A} \rightarrow \mathcal{B}$. We must prove that $\mathcal{B}^* \in \mathbf{H}(\mathcal{A}^*)$. It suffices to consider the following homomorphism:

$g : \mathcal{A}^* \rightarrow \mathcal{B}^*$ defined by:

$$g(a) = \begin{cases} h(a) & \text{if } a \in (\mathcal{A}^*)_+ \\ \neg h(\neg a) & \text{if } a \in (\mathcal{A}^*)_- \end{cases}$$

Take now $\mathcal{B} \in \mathbf{H}(\mathbb{K}^*)$, i.e., \mathcal{B} is the image of some homomorphism $h : \mathcal{A}^* \rightarrow \mathcal{B}$, where $\mathcal{A} \in \mathbb{K}$. Then $\mathcal{B} \cong h[(\mathcal{A}^*)_+]^*$, so $\mathcal{B} \in \mathbf{H}(\mathcal{A})^* \subseteq \mathbf{H}(\mathbb{K})^*$. \square

Lemma 6.31. *If \mathbb{K} is a class of prelinear semihoops, then $\mathbf{S}(\mathbb{K}^*) = \mathbf{S}(\mathbb{K})^*$.*

Proof: Take $\mathcal{A} \in \mathbb{K}$ and $\mathcal{B} \in \mathbf{S}(\mathcal{A}^*)$. Since \mathcal{B} is a subalgebra of the disconnected rotation of \mathcal{A} , we have that $B_+ \subseteq (A^*)_+ = A$ and $B_- \subseteq (A^*)_-$. Actually, B_+ is the universe of a subalgebra of \mathcal{A} and \mathcal{B} is the disconnected rotation of this subalgebra. Therefore, $\mathcal{A} \in \mathbf{S}(\mathbb{K})^*$.

Conversely, if $\mathcal{A} \in \mathbb{K}$ and $\mathcal{B} \in \mathbf{S}(\mathcal{A})^*$, then $\mathcal{B} = \mathcal{C}^*$ for some $\mathcal{C} \subseteq \mathcal{A}$. It follows that $\mathcal{C}^* \subseteq \mathcal{A}^*$, so $\mathcal{B} \in \mathbf{S}(\mathcal{A}^*) \subseteq \mathbf{S}(\mathbb{K}^*)$. \square

Lemma 6.32. *If \mathbb{K} is a class of prelinear semihoops, then $\mathbf{P}_U(\mathbb{K})^* \subseteq \mathbf{ISP}_U(\mathbb{K}^*)$.*

Proof: Take $\{\mathcal{A}_i : i \in I\} \subseteq \mathbb{K}$ and consider an ultraproduct $\prod_{\mathcal{U}}^I \mathcal{A}_i$. Consider also the ultraproduct of $\{\mathcal{A}_i^* : i \in I\}$ corresponding to the same index set and the same ultrafilter, i.e., $\prod_{\mathcal{U}}^I \mathcal{A}_i^*$. It suffices to take the embedding $\alpha : (\prod_{\mathcal{U}}^I \mathcal{A}_i)^* \rightarrow \prod_{\mathcal{U}}^I \mathcal{A}_i^*$ defined by:

- If $\bar{a}/\mathcal{U} \in (\prod_{\mathcal{U}}^I \mathcal{A}_i)_+^*$, $\alpha(\bar{a}/\mathcal{U}) := \bar{a}/\mathcal{U}$.
- If $\neg(\bar{a}/\mathcal{U}) \in (\prod_{\mathcal{U}}^I \mathcal{A}_i)_-^*$, $\alpha(\neg(\bar{a}/\mathcal{U})) := \neg\bar{a}/\mathcal{U}$, where for every $i \in I$ $(\neg\bar{a})(i) = \neg(\bar{a}(i))$.

We obtain $(\prod_{\mathcal{U}}^I \mathcal{A}_i)^* \in \mathbf{IS}(\prod_{\mathcal{U}}^I \mathcal{A}_i^*) \subseteq \mathbf{ISP}_U(\mathbb{K}^*)$. \square

Lemma 6.33. *If \mathbb{K} is a class of prelinear semihoops, then $\mathbf{P}_U(\mathbb{K}^*) \subseteq \mathbf{IS}(\mathbf{P}_U(\mathbb{K})^*)$.*

Proof: Take $\{\mathcal{A}_i : i \in I\} \subseteq \mathbb{K}$ and an ultraproduct $\prod_{\mathcal{U}}^I \mathcal{A}_i^*$. Given $\bar{a} \in \prod_{\mathcal{U}}^I \mathcal{A}_i^*$, we define $j(\bar{a}) \in \prod_{\mathcal{U}}^I \mathcal{A}_i$ as:

$$j(\bar{a})(i) := \begin{cases} \bar{a}(i) & \text{if } \bar{a}(i) > \neg\bar{a}(i), \\ \neg\bar{a}(i) & \text{otherwise.} \end{cases}$$

In order to show that $\prod_{\mathcal{U}}^I \mathcal{A}_i^* \in \mathbf{IS}((\prod_{\mathcal{U}}^I \mathcal{A}_i)^*) \subseteq \mathbf{IS}(\mathbf{P}_U(\mathbb{K})^*)$ it is enough to consider the embedding $\alpha : \prod_{\mathcal{U}}^I \mathcal{A}_i^* \rightarrow (\prod_{\mathcal{U}}^I \mathcal{A}_i)^*$ defined by:

- If $\bar{a}/\mathcal{U} \in \prod_{\mathcal{U}}^I \mathcal{A}_i^*$ is such that $\{i \in I : \bar{a}(i) > \neg\bar{a}(i)\} \in \mathcal{U}$, then $\alpha(\bar{a}/\mathcal{U}) := j(\bar{a})/\mathcal{U}$.
- If $\bar{a}/\mathcal{U} \in \prod_{\mathcal{U}}^I \mathcal{A}_i^*$ is such that $\{i \in I : \bar{a}(i) < \neg\bar{a}(i)\} \in \mathcal{U}$, then $\alpha(\bar{a}/\mathcal{U}) := \neg\alpha(\neg\bar{a}/\mathcal{U})$.

\square

Theorem 6.34. *Let \mathbb{K} and \mathbb{L} be classes of totally ordered prelinear semihoops. Then:*

- $\mathbf{V}(\mathbb{K}) \subseteq \mathbf{V}(\mathbb{L})$ if, and only if, $\mathbf{V}(\mathbb{K}^*) \subseteq \mathbf{V}(\mathbb{L}^*)$, and

- $\mathbf{V}(\mathbb{K}) = \mathbf{V}(\mathbb{L})$ if, and only if, $\mathbf{V}(\mathbb{K}^*) = \mathbf{V}(\mathbb{L}^*)$.

Proof: From Jónsson's Lemma (see [24]) we deduce that given a class \mathbb{M} of prelinear semihoops or IMTL-algebras, $\mathbf{HSP}_U(\mathbb{M})$ coincides with the class of the chains in $\mathbf{V}(\mathbb{M})$. Therefore, due to the representation in subdirect products of chains, we only need to prove: $\mathbf{HSP}_U(\mathbb{K}) \subseteq \mathbf{HSP}_U(\mathbb{L})$ iff $\mathbf{HSP}_U(\mathbb{K}^*) \subseteq \mathbf{HSP}_U(\mathbb{L}^*)$.

Suppose first that $\mathbf{HSP}_U(\mathbb{K}) \subseteq \mathbf{HSP}_U(\mathbb{L})$. Therefore, $\mathbb{K} \subseteq \mathbf{HSP}_U(\mathbb{L})$, so $\mathbb{K}^* \subseteq (\mathbf{HSP}_U(\mathbb{L}))^* = \mathbf{HS}(\mathbf{P}_U(\mathbb{L}))^* \subseteq \mathbf{HSISP}_U(\mathbb{L}^*) \subseteq \mathbf{HSP}_U(\mathbb{L}^*)$. Thus, $\mathbf{HSP}_U(\mathbb{K}^*) \subseteq \mathbf{HSP}_U(\mathbb{L}^*)$.

Suppose now that $\mathbf{HSP}_U(\mathbb{K}^*) \subseteq \mathbf{HSP}_U(\mathbb{L}^*)$. Then, $\mathbb{K}^* \subseteq \mathbf{HSP}_U(\mathbb{L}^*) \subseteq \mathbf{HSISP}_U(\mathbf{P}_U(\mathbb{L}))^* = (\mathbf{HSISP}_U(\mathbb{L}))^* \subseteq (\mathbf{HSP}_U(\mathbb{L}))^*$. Therefore, $\mathbb{K} \subseteq \mathbf{HSP}_U(\mathbb{L})$.

The other equivalence trivially follows from this one. \square

Corollary 6.35. *Given any variety of prelinear semihoops \mathbb{K} , define $\sigma(\mathbb{K}) := \mathbf{V}(\{\text{chains of } \mathbb{K}\}^*)$. Then, σ is an isomorphism between the lattice of varieties of prelinear semihoops and the lattice of subvarieties of \mathbb{IBP}_0 .*

6.5 Adding the fixpoint to perfect MTL-algebras. Logics \mathbf{BP}^{+n} and $\mathbf{BP}^{+\omega}$. Glivenko theorems, standard completeness and other properties

In this section we will use perfect MTL-algebras to construct new kinds of MTL-algebras and we will study the varieties and the logics that they define.

First, we extend the construction of connected rotation-annihilation (see Chapter 4) in such a way that it will not only produce IMTL-algebras, but also non-involutive MTL-algebras.

Definition 6.36. *Let \mathcal{A} and \mathcal{B} be MTL-algebras such that \mathcal{A} is perfect. We define a new MTL-algebra \mathcal{C} , whose carrier is $A \cup (B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\})$, the orderings in A and B are extended by letting $a < b < c$ for every $a \in A_-$, $b \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\}$ and $c \in A_+$, and the operations are defined as $\bar{0}^{\mathcal{C}} := \bar{0}^{\mathcal{A}}$, $\bar{1}^{\mathcal{C}} := \bar{1}^{\mathcal{A}}$ and:*

$$a \&^{\mathcal{C}} b := \begin{cases} a \&^{\mathcal{A}} b & \text{if } a, b \in A \\ \bar{0}^{\mathcal{A}} & \text{if } a, b \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\} \text{ and } a \leq \neg^{\mathcal{B}} b \\ a \&^{\mathcal{B}} b & \text{if } a, b \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\} \text{ and } a \not\leq \neg^{\mathcal{B}} b \\ b & \text{if } a \in A_+, b \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\} \\ a & \text{if } a \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\}, b \in A_+ \\ \bar{0}^{\mathcal{A}} & \text{if } a \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\}, b \in A_- \\ \bar{0}^{\mathcal{A}} & \text{if } a \in A_-, b \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\} \end{cases}$$

$$a \rightarrow^{\mathcal{C}} b := \begin{cases} a \rightarrow^{\mathcal{A}} b & \text{if } a, b \in A \\ \bar{1}^{\mathcal{C}} & \text{if } a, b \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\}, a \leq b \\ a \rightarrow^{\mathcal{B}} b & \text{if } a, b \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\}, a \not\leq b \\ b & \text{if } a \in A_+, b \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\} \\ \bar{1}^{\mathcal{A}} & \text{if } a \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\}, b \in A_+ \\ \neg^{\mathcal{B}} a & \text{if } a \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\}, b \in A_- \\ \bar{1}^{\mathcal{A}} & \text{if } a \in A_-, b \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\} \end{cases}$$

It is routine to check that \mathcal{C} is indeed an MTL-algebra. Since this is a generalization of the connected rotation-annihilation construction, we keep on denoting it as $\mathcal{A} \odot \mathcal{B}$. Therefore, we can also speak about decompositions of MTL-chains in rotation-annihilation of a perfect MTL-chain and another MTL-chain. Moreover, propositions 4.65 and 4.66 can be easily generalized to this more general case. We will use this construction in some special cases, but first we need to introduce a particular kind of WNM-chains.

Definition 6.37. For every natural number $n \geq 1$, we define a WNM-chain $\mathcal{W}_n = \langle W_n, \&, \rightarrow, \wedge, \vee, \bar{0}^{\mathcal{W}_n}, \bar{1}^{\mathcal{W}_n} \rangle$ by taking $W_n = \{\bar{1}^{\mathcal{W}_n} > a_0 > a_1 > \dots > a_{n-1} > \bar{0}^{\mathcal{W}_n}\}$ and $\neg a_i = a_0$ for every $i < n$. As in every WNM-chain, the operations $\&$ and \rightarrow are defined as:

$$a \& b := \begin{cases} a \wedge b & \text{if } a > \neg b, \\ \bar{0}^{\mathcal{W}_n} & \text{otherwise.} \end{cases}$$

$$a \rightarrow b := \begin{cases} \bar{1}^{\mathcal{W}_n} & \text{if } a \leq b, \\ \neg a \vee b & \text{otherwise.} \end{cases}$$

for every $a, b \in W_n$.

Moreover, we define the WNM-chain $\mathcal{W}_\omega = \langle W_\omega, \&, \rightarrow, \wedge, \vee, \bar{0}^{\mathcal{W}_\omega}, \bar{1}^{\mathcal{W}_\omega} \rangle$ by taking an infinite set $\{a_k : k < \omega\}$, letting $W_\omega = \{\bar{1}^{\mathcal{W}_\omega} > a_0 > a_1 > \dots > a_n > a_{n+1} > \dots > \bar{0}^{\mathcal{W}_\omega}\}$ and defining the operations in the same way.

Notice that in these chains the product of any pair of elements different from the top, is always the bottom. This is sometimes called the *drastic product*. Moreover, all of them have negation fixpoint, namely a_0 . It is clear that $\mathbf{L}_3 \cong \mathcal{W}_1$.

Definition 6.38. Let \mathcal{A} be a perfect MTL-algebra and $1 \leq n \leq \omega$, $\mathcal{A} \odot \mathcal{W}_n$ is denoted as \mathcal{A}^{+n} and we call it a perfect algebra plus n points. Since a_0 is a negation fixpoint, when $n = 1$ we call \mathcal{A}^{+1} a perfect algebra plus fixpoint.

We must be careful to avoid any misunderstanding here. Perfect algebras cannot have fixpoint. Therefore, we are not saying that \mathcal{A}^{+n} is a perfect algebra with fixpoint; this would not make sense. On the contrary, we are just saying that \mathcal{A}^{+n} is a perfect algebra plus n points, in the sense that it is obtained by adding n new points to a given perfect algebra \mathcal{A} . Thus, \mathcal{A}^{+n} is not perfect and it has negation fixpoint.

Notice that if we start with an IMTL-algebra, this definition is only preserving the involution when $n = 1$. Moreover, the construction of \mathcal{A}^{+1} is canonical in the sense that it is the only possible way to add the fixpoint to a perfect algebra:

Theorem 6.39. *Let \mathcal{A} be an MTL-algebra with negation fixpoint such that $A = A_+ \cup A_-$ and $\text{Rad}(\mathcal{A}) = A_+$. Let a be the fixpoint. Then, $a \& b = a$ for every $b \in A_+$ and $a \& b = \bar{0}^{\mathcal{A}}$ for every $b \in A_-$.*

Proof: Take $b > a$. We know that $a \& b \leq a$. Suppose $a \& b < a$. Then, $\neg(a \& b) \in A_+$, hence $b \& \neg(a \& b) \in A_+$. This implies $a > \neg(b \& \neg(a \& b))$, in contradiction with $a \& (b \& \neg(a \& b)) = (a \& b) \& \neg(a \& b) = \bar{0}^{\mathcal{A}}$. If $b \in A_-$, then $\neg b \in A_+$, so $a \leq \neg b$ and this is equivalent to $a \& b = \bar{0}^{\mathcal{A}}$. \square

The class of perfect IMTL-algebras plus fixpoint coincides with the class of all connected rotations of MTL-algebras without zero divisors:

Theorem 6.40. *Let \mathcal{A} be an IMTL-algebra. The following are equivalent:*

- (1) \mathcal{A} is a perfect algebra plus fixpoint.
- (2) \mathcal{A} is isomorphic to the connected rotation of an MTL-algebra without zero divisors.

Proof: (1) \Rightarrow (2): Let a be the fixpoint of the algebra. Consider the MTL-algebra \mathcal{B} defined by $\text{Rad}(\mathcal{A}) \cup \{a\}$ such that $\bar{0}^{\mathcal{B}} = a$. Since the radical is closed under $\&$, \mathcal{B} is an MTL-algebra without zero divisors. Thus $\mathcal{A} \cong \mathcal{B}^*$.

(2) \Rightarrow (1): If $\mathcal{A} \cong \mathcal{B}^*$ for some MTL-algebra \mathcal{B} without zero divisors, then is clear that all the positive elements have infinite order and all the negative elements have finite order, so it is a perfect algebra plus the fixpoint. \square

Proposition 6.41. *Let \mathcal{A} be a perfect MTL-algebra and n any ordinal number such that $1 \leq n \leq \omega$. Then, $\mathcal{A}^{+n}/\text{Rad}(\mathcal{A}^{+n}) \cong \mathcal{W}_n$.*

Proof: Recall that $\text{Rad}(\mathcal{A}^{+n}) = A_+$. So, on one hand, it is clear that $\bar{1}^{\mathcal{A}^{+n}}/\text{Rad}(\mathcal{A}^{+n}) = A_+$ and $\bar{0}^{\mathcal{A}^{+n}}/\text{Rad}(\mathcal{A}^{+n}) = A_-$. On the other hand, for every $a, b \in W_n \setminus \{\bar{0}^{\mathcal{W}_n}, \bar{1}^{\mathcal{W}_n}\}$ such that $a < b$, we have $b \rightarrow a = a_0 \notin \text{Rad}(\mathcal{A}^{+n})$, thus $a/\text{Rad}(\mathcal{A}^{+n}) \neq b/\text{Rad}(\mathcal{A}^{+n})$. Therefore, the function defined by:

$$f(x/\text{Rad}(\mathcal{A}^{+n})) := \begin{cases} \bar{1}^{\mathcal{W}_n} & \text{if } x = \bar{1}^{\mathcal{A}^{+n}} \\ \bar{0}^{\mathcal{W}_n} & \text{if } x = \bar{0}^{\mathcal{A}^{+n}} \\ x & \text{if } x \in W_n \setminus \{\bar{0}^{\mathcal{W}_n}, \bar{1}^{\mathcal{W}_n}\} \end{cases}$$

is an isomorphism from $\mathcal{A}^{+n}/\text{Rad}(\mathcal{A}^{+n})$ to \mathcal{W}_n . \square

However, in this case the quotient by the radical does not characterize perfect algebras plus fixpoint. This is false even for MV-algebras. Take for instance the MV-algebra \mathbf{L}_3^ω . Indeed, $\mathbf{L}_3^\omega/\text{Rad}(\mathbf{L}_3^\omega) \cong \mathcal{W}_1$ but \mathbf{L}_3^ω is not a perfect algebra plus fixpoint.

Definition 6.42. For every $1 \leq n \leq \omega$, let \mathbb{BP}_0^{+n} be the variety generated by all perfect MTL-algebras plus n points, and define $\mathbb{IBP}_0^{+1} := \mathbb{BP}_0^{+1} \cap \mathbb{IMTL}$.

Obviously, $\mathbb{IBP}_0 \subsetneq \mathbb{IBP}_0^{+1}$, since $\mathbf{L}_3 \in \mathbb{IBP}_0^{+1} \setminus \mathbb{IBP}_0$.

It is also clear that we have the following chain of strict inclusions:

$$\mathbb{BP}_0 \subsetneq \mathbb{BP}_0^{+1} \subsetneq \dots \subsetneq \mathbb{BP}_0^{+n} \subsetneq \mathbb{BP}_0^{+(n+1)} \subsetneq \dots \subsetneq \mathbb{BP}_0^{+\omega}.$$

Proposition 6.43. $\mathbb{BP}_0^{+\omega}$ is the minimum variety containing \mathbb{BP}_0^{+n} for every finite n , i.e. $\mathbb{BP}_0^{+\omega} = \bigvee_{1 \leq n < \omega} \mathbb{BP}_0^{+n}$.

Proof: It is obvious that $\bigvee_{1 \leq n < \omega} \mathbb{BP}_0^{+n} \subseteq \mathbb{BP}_0^{+\omega}$. To prove the other inclusion, consider any equation $\varphi \approx \psi \in Eq_{\mathcal{L}}$ such that is not verified by all algebras in $\mathbb{BP}_0^{+\omega}$. We must show that $\varphi \approx \psi$ is not verified by all algebras in $\bigvee_{1 \leq n < \omega} \mathbb{BP}_0^{+n}$. Suppose that $\{x_1, \dots, x_n\}$ is the set of variables appearing in $\varphi \approx \psi$. There is a chain $\mathcal{C} \in \mathbb{BP}_0^{+\omega}$ and an evaluation v in \mathcal{C} such that $v(\varphi) \neq v(\psi)$. If \mathcal{C} is perfect or $\mathcal{C} = \mathcal{A}^{+k}$ for some $k < \omega$ and some perfect algebra \mathcal{A} , the proof finishes. Suppose that $\mathcal{C} = \mathcal{A}^{+\omega}$ for some perfect algebra \mathcal{A} . Then the subalgebra generated by the set $A_+ \cup \overline{A_+} \cup \{v(x_1), \dots, v(x_n)\}$ is also not satisfying the equation and it belongs to the variety $\mathbb{BP}_0^{+(n+1)}$. \square

Theorem 6.44. We can obtain an equational base for $\mathbb{BP}_0^{+\omega}$ by adding to the axioms of MTL the following:

1. $Bp(x) \vee (\neg x \leftrightarrow \neg \neg x) \approx \bar{1}$
2. $(x \vee \neg x \rightarrow y \vee \neg y) \vee ((y \vee \neg y \rightarrow \neg y \wedge \neg \neg y) \rightarrow y \vee \neg y) \vee (((x \vee \neg x)^2 \rightarrow y \vee \neg y) \rightarrow y \vee \neg y) \approx \bar{1}$
3. $Bp(x) \vee (\neg y \leftrightarrow \neg \neg y) \vee (x \rightarrow x \& p(y)) \approx \bar{1}$

Proof: Let \mathbb{K} be the variety of MTL-algebras where these equations are valid. Let \mathcal{A} be a perfect MTL-algebra plus ω points. One can easily check that $\mathcal{A} \models ((\neg(\neg x)^2)^2 \leftrightarrow \neg(\neg x^2)^2) \vee (\neg x \leftrightarrow \neg \neg x) \approx \bar{1}$. Let us prove that also the second equation is valid in \mathcal{A} . Take $a, b \in A$. If $\neg a$ is the fixpoint, then $a \vee \neg a \rightarrow b \vee \neg b = \bar{1}^A$ and the equation is satisfied. Suppose now that $\neg b$ is the fixpoint and $\neg a \neq \neg \neg a$. $a \vee \neg a > b \vee \neg b$, so $(a \vee \neg a)^2 > b \vee \neg b$. Thus, $(a \vee \neg a)^2 \rightarrow b \vee \neg b = b \vee \neg b$ and the equation is satisfied too. Finally, suppose that neither $\neg a$ nor $\neg b$ are the fixpoint. Then $b \vee \neg b \rightarrow \neg b \wedge \neg \neg b \in A_-$, hence $(b \vee \neg b \rightarrow \neg b \wedge \neg \neg b) \rightarrow b \vee \neg b = \bar{1}^A$. Finally, let us prove that also the third equation is valid in \mathcal{A} . Take $a, b \in A$. Suppose $Bp(a) \neq \bar{1}^A$ and $\neg \neg b \neq \neg b$ (otherwise the equation is clearly satisfied). Then, we have $a \notin \text{Rad}(\mathcal{A}) \cup \text{Rad}(\overline{\mathcal{A}})$ and $p(b) \in A_+$, so $a \& p(b) = a$ and the equation is also satisfied. Therefore, $\mathbb{BP}_0^{+\omega} \subseteq \mathbb{K}$.

In order to prove the other inclusion and taking into account the representation theorem in subdirect products of chains, we only need to check that all chains in \mathbb{K} are either perfect or perfect plus some points. Let \mathcal{C} be such a chain and take $a \in C_+$; we will see that $a^2 \in C_+$. Suppose that it is not

true. Then there are two possibilities: either a^2 is the fixpoint or it is smaller than its negation. If $a^2 = b = \neg b$, then the second equation would imply $(a \rightarrow b) \vee ((b \rightarrow b) \rightarrow b) \vee ((a^2 \rightarrow b) \rightarrow b) = (a \rightarrow b) \vee b \vee b = a \rightarrow b = \bar{1}^A$, so $a \leq b$, a contradiction. Suppose now, that $a^2 < \neg a^2$. By the first equation $\neg(\neg a^2)^2 = (\neg(\neg a)^2)^2 = (\neg \bar{0}^A)^2 = \bar{1}^A$, so $(\neg a^2)^2 = \bar{0}^A$, i. e. $\neg a^2 \leq \neg \neg a^2$. This means that $a^2 < \neg a^2 \leq \neg \neg a^2$, so $\neg a^2 = \neg \neg a^2$. Therefore $\neg a^2$ is the fixpoint. Using values a and $\neg a^2$ in the second equation we obtain: $(a \rightarrow \neg a^2) \vee ((\neg a^2 \rightarrow \neg a^2) \rightarrow \neg a^2) \vee ((a^2 \rightarrow \neg a^2) \rightarrow \neg a^2) = (a \rightarrow \neg a^2) \vee \neg a^2 \vee ((a^2 \rightarrow \neg a^2) \rightarrow \neg a^2) = (a \rightarrow \neg a^2) \vee ((a^2 \rightarrow \neg a^2) \rightarrow \neg a^2) = \bar{1}^A$, so one of the two disjuncts must be $\bar{1}^A$. $a > \neg a^2$, thus $a^2 \rightarrow \neg a^2 \leq \neg a^2$, but this is absurd since $a^2 \rightarrow \neg a^2 = \bar{1}^A$. Thus, given $a, b \in C_+$ such that $a \leq b$, we have $a \& b \geq a^2 \in C_+$; therefore, C_+ is closed under $\&$. If $C = \text{Rad}(\mathcal{C}) \cup \overline{\text{Rad}(\mathcal{C})}$, the chain is perfect. Suppose not. Then for every $a \notin \text{Rad}(\mathcal{C}) \cup \overline{\text{Rad}(\mathcal{C})}$, we have $\neg a = \neg \neg a$. Indeed, $a \leq \neg a$ (because $a \notin \text{Rad}(\mathcal{C}) = C_+$), and $\neg a \leq \neg \neg a$ (because $\neg a \notin \text{Rad}(\mathcal{C}) = C_+$). Thus, $a \leq \neg a \leq \neg \neg a$, and this implies $\neg a = \neg \neg a$. Moreover, given $a \notin \text{Rad}(\mathcal{C}) \cup \overline{\text{Rad}(\mathcal{C})}$, and $b \in C_+$, the third equation implies $a \& b = b$, so \mathcal{C} is perfect plus some points. \square

Theorem 6.45. *If $1 \leq n < \omega$, we can obtain an equational base for \mathbb{BP}_0^{+n} by adding to the axioms of MTL the following:*

1. $Bp(x) \vee (\neg x \leftrightarrow \neg \neg x) \approx \bar{1}$
2. $(x \vee \neg x \rightarrow y \vee \neg y) \vee ((y \vee \neg y \rightarrow \neg y \wedge \neg \neg y) \rightarrow y \vee \neg y) \vee (((x \vee \neg x)^2 \rightarrow y \vee \neg y) \rightarrow y \vee \neg y) \approx \bar{1}$
3. $Bp(x) \vee (\neg y \leftrightarrow \neg \neg y) \vee (x \rightarrow x \& p(y)) \approx \bar{1}$
4. $\bigvee_{0 \leq i \leq n} Bp(x_i) \vee \bigvee_{0 \leq i < j \leq n} (x_i \leftrightarrow x_j) \approx \bar{1}$

Proof: Let \mathbb{K} be the variety defined by these equations. Let \mathcal{A}^{+n} be a perfect algebra plus n points. By the previous theorem the first three equations are valid in this algebra. Let us check the fourth one. Consider $a_0, \dots, a_n \in \mathcal{A}^{+n}$. If there is some i such that $a_i \in \text{Rad}(\mathcal{A}^{+n}) \cup \overline{\text{Rad}(\mathcal{A}^{+n})}$, then $Bp(a_i) = \bar{1}^{\mathcal{A}^{+n}}$. If for every i $a_i \notin \text{Rad}(\mathcal{A}^{+n}) \cup \overline{\text{Rad}(\mathcal{A}^{+n})}$, then there must be some i, j such that $a_i = a_j$, since there are only n elements in these conditions, so $a_i \leftrightarrow a_j = \bar{1}^{\mathcal{A}^{+n}}$ and the equation is also satisfied. Therefore, $\mathbb{BP}_0^{+n} \subseteq \mathbb{K}$. Conversely, take any chain $\mathcal{C} \in \mathbb{K}$. On one hand, by the proof of the previous theorem we know that \mathcal{C} is perfect or perfect plus some points. On the other hand, the fourth equation implies that there are at most n points not belonging to $\text{Rad}(\mathcal{C}) \cup \overline{\text{Rad}(\mathcal{C})}$. Therefore, we obtain $\mathcal{C} \in \mathbb{BP}_0^{+n}$, hence $\mathbb{K} \subseteq \mathbb{BP}_0^{+n}$. \square

Corollary 6.46. *An equational base for \mathbb{IBP}_0^{+1} is obtained by adding to the axioms of IMTL the following:*

1. $Bp(x) \vee (x \leftrightarrow \neg x) \approx \bar{1}$

$$2. (x \vee \neg x \rightarrow y \vee \neg y) \vee ((y \vee \neg y \rightarrow y \wedge \neg y) \rightarrow y \vee \neg y) \vee (((x \vee \neg x)^2 \rightarrow y \vee \neg y) \rightarrow y \vee \neg y) \approx \bar{1}$$

Notice that the equation $Bp(x) \vee (\neg y \leftrightarrow \neg \neg y) \vee (x \rightarrow x \& p(y)) \approx \bar{1}$ of the last two theorems is strictly necessary. Indeed, if \mathcal{A} is any perfect MTL-algebra, we can define an MTL-algebra \mathcal{B} whose carrier is $A \cup \{a, b\}$, the ordering in A is extended by letting $x < b < a < y$ for every $x \in A_-$, $y \in A_+$, and the operations are defined as $\bar{0}^{\mathcal{B}} = \bar{0}^{\mathcal{A}}$, $\bar{1}^{\mathcal{B}} = \bar{1}^{\mathcal{A}}$ and:

$$x \&^{\mathcal{B}} y := \begin{cases} x \&^{\mathcal{A}} y & \text{if } x, y \in A \\ \bar{0}^{\mathcal{A}} & \text{if } x, y \in \{a, b\} \\ b & \text{if } x \in A_+, y \in \{a, b\} \\ \bar{0}^{\mathcal{A}} & \text{if } x \in A_-, y \in \{a, b\} \end{cases}$$

and \rightarrow is its residuum. Then, \mathcal{B} is neither in $\mathbb{BP}_0^{+\omega}$ nor in \mathbb{BP}_0^{+2} , and it does not satisfy the equation $Bp(x) \vee (\neg y \leftrightarrow \neg \neg y) \vee (x \rightarrow x \& p(y)) \approx \bar{1}$, even though it satisfies the remaining equations.

If \mathbb{BP}_0^{+1} , \mathbb{IBP}_0^{+1} , \mathbb{BP}_0^{+n} and \mathbf{W}_n are respectively the logics associated to the varieties \mathbb{BP}_0^{+1} , \mathbb{IBP}_0^{+1} , \mathbb{BP}_0^{+n} and $\mathbf{V}(\mathcal{W}_n)$ for every $2 \leq n \leq \omega$, and \mathbf{L}_3 is the three-valued logic of Łukasiewicz, i.e. the logic associated to the variety $\mathbf{V}(\mathbf{L}_3)$, we can prove the following Glivenko-style theorems for these logics:

Theorem 6.47. *For every $\varphi \in Fm_{\mathcal{L}}$, we have:*

- (i) $\vdash_{\mathbf{L}_3} \varphi$ if, and only if, $\vdash_{\mathbb{BP}_0^{+1}} t(\varphi) \vee (t(\varphi \leftrightarrow \neg \varphi) \& \varphi)$.
- (ii) $\vdash_{\mathbf{L}_3} \varphi$ if, and only if, $\vdash_{\mathbb{IBP}_0^{+1}} t(\varphi) \vee (t(\varphi \leftrightarrow \neg \varphi) \& \varphi)$.
- (iii) $\vdash_{\mathbf{W}_n} \varphi$ if, and only if, $\vdash_{\mathbb{BP}_0^{+n}} t(\varphi) \vee (t(\neg \varphi \leftrightarrow \neg \neg \varphi) \& \varphi)$, for every $2 \leq n \leq \omega$.

$$\text{where } t(x) = \neg(\neg x^2)^2.$$

Proof: We will prove the first case as an example. The remaining ones are analogous. Suppose that $\vdash_{\mathbf{L}_3} \varphi$ and take any chain $\mathcal{A} \in \mathbb{BP}_0^{+1}$. We must prove that $\mathcal{A} \models t(\varphi) \vee t(\varphi \leftrightarrow \neg \varphi) \& \varphi \approx \bar{1}$. Let $v : Fm_{\mathcal{L}} \rightarrow \mathcal{A}$ be an evaluation. We know that $\mathcal{A}/\text{Rad}(\mathcal{A}) \cong \mathbf{L}_3$, so $v(\varphi)/\text{Rad}(\mathcal{A}) = \bar{1}^{\mathcal{A}}/\text{Rad}(\mathcal{A})$, i.e. $v(\varphi) \in A_+ = \text{Rad}(\mathcal{A})$, hence $t(v(\varphi)) = \bar{1}^{\mathcal{A}}$. Conversely, if $\vdash_{\mathbb{BP}_0^{+1}} t(\varphi) \vee (t(\varphi \leftrightarrow \neg \varphi) \& \varphi)$, then in particular $\mathbf{L}_3 \models t(\varphi) \vee (t(\varphi \leftrightarrow \neg \varphi) \& \varphi) \approx \bar{1}$. Let v be any evaluation on \mathbf{L}_3 . We have $t(v(\varphi)) \vee (t(v(\varphi) \leftrightarrow \neg v(\varphi)) \& v(\varphi)) = 1$. The assumptions $v(\varphi) = 0$ and $v(\varphi) = \frac{1}{2}$ lead to contradiction, so it must be $v(\varphi) = 1$, and this finishes the proof. \square

Finally, we will discuss which of those varieties define new fuzzy logics with strong standard completeness theorem. We will prove the theorem for $\mathbb{BP}_0^{+\omega}$, \mathbb{BP}_0 , \mathbb{IBP}_0^{+1} and \mathbb{BP}_0^{+1} . For $\mathbb{BP}_0^{+\omega}$ the original method of Jenei and Montagna [100], that we have sketched in Chapter 5, will be enough to prove it, while for

\mathbb{BP}_0 , \mathbb{IBP}_0^{+1} and \mathbb{BP}_0^{+1} we will need some modifications of the method. For the remaining varieties, \mathbb{BP}_0^{+n} (for every $1 < n < \omega$) and \mathbb{IBP}_0 we will prove that there is no standard completeness.

Theorem 6.48. *The logic $\mathbb{BP}_0^{+\omega}$ enjoys the SSC.*

Proof: Let $\mathcal{A} \in \mathbb{BP}_0^{+\omega}$ be a countable perfect chain plus infinitely many points. Using the method of Jenei and Montagna we obtain an MTL-chain \mathcal{B} over $[0, 1]$ and an embedding $h : \mathcal{A} \rightarrow \mathcal{B}$. It is easy to check that $\text{Rad}(\mathcal{B}) = B_+$, so $\mathcal{B} \in \mathbb{BP}_0^{+\omega}$. \square

Theorem 6.49. *The logic \mathbb{BP}_0 enjoys the SSC.*

Proof: Let $\mathcal{A} \in \mathbb{BP}_0$ be a countable chain. We know that \mathcal{A} is perfect. If A_- has no maximum, the method of Jenei and Montagna would not work. Indeed, the resulting chain over $[0, 1]$ would have a negation fixpoint, so it would not belong to \mathbb{BP}_0 . To avoid this problem and make sure that A_- has a maximum element, we add a couple of new elements $a, b \notin A$ requiring:

- $a < x$ for each $x \in A_+$,
- $b < a$,
- $x < b$ for each $x \in A_-$,
- $\neg a = b$,
- $\neg b = a$,
- $a \& x = x \& a = a$ for each $x \in A_+ \cup \{a\}$,
- $a \& x = x \& a = \bar{0}^A$ for each $x \in A_- \cup \{b\}$,
- $b \& x = x \& b = b$ for each $x \in A_+$ and
- $b \& x = x \& b = \bar{0}^A$ for each $x \in A_- \cup \{b\}$.

\mathcal{A} is a subalgebra of this extended chain. Therefore, we can suppose without losing generality that A_- has a maximum, say b . Now we apply the usual method of Jenei and Montagna. First we obtain a densely ordered countable monoid \mathcal{B} over the set $\{\langle \bar{0}^A, 1 \rangle\} \cup \{\langle a, q \rangle : a \in A \setminus \{\bar{0}^A\}, q \in Q \cap (0, 1]\}$, with the lexicographical order and the following monoidal operation:

$$\langle a, q \rangle \circ \langle c, r \rangle := \begin{cases} \min\{\langle a, q \rangle, \langle c, r \rangle\} & \text{if } a \& c = \min\{a, c\} \\ \langle a \& c, 1 \rangle & \text{otherwise.} \end{cases}$$

Notice that for every $\langle c, r \rangle > \langle b, 1 \rangle$ (i.e. $c > b$) we have $\langle c, r \rangle^n > \langle b, 1 \rangle$ for every $n \geq 1$. Notice also that given $\langle c, r \rangle \leq \langle b, 1 \rangle$ we can define $\neg \langle c, r \rangle := \max\{\langle a, q \rangle : \langle a, q \rangle \circ \langle c, r \rangle = \langle \bar{0}^A, 1 \rangle\}$ and we get $\neg \langle c, r \rangle > \langle b, 1 \rangle$.

\mathcal{B} is isomorphic to a monoid over $Q \cap [0, 1]$ and it is completed to $[0, 1]$ by defining $\alpha \otimes \beta := \sup\{p \circ q : p, q \in Q, p \leq \alpha, q \leq \beta\}$ and we obtain an MTL-chain \mathcal{C} over $[0, 1]$ and an embedding $h : \mathcal{A} \rightarrow \mathcal{C}$. It is easy to check that \mathcal{C} is perfect. \square

Theorem 6.50. *The logic IBP_0^{+1} enjoys the SSC.*

Proof: Let $\mathcal{A} \in \text{IBP}_0^{+1}$ be a countable chain. As we have seen, \mathcal{A} is either perfect or perfect plus fixpoint. It is enough to suppose that \mathcal{A} is a countable perfect chain plus fixpoint and show that it can be embedded in a standard chain of IBP_0^{+1} over $[0, 1]$. Let $a \in A$ be the fixpoint. If we use the usual method we first obtain an algebra over a densely ordered set B , as we have described in the preliminaries. For every $q \in Q \cap (0, 1)$ the element $\langle a, q \rangle \in B$ is such that $\neg\langle a, q \rangle = \langle a, 1 \rangle$, so the resulting standard chain will not be perfect plus fixpoint.

In order to solve this problem, we consider the construction of Jenei and Montagna applied to the prelinear semihoop defined by $\text{Rad}(\mathcal{A})$, but giving an algebra \mathcal{C} over $[0.6, 1]$ instead of being over $[0, 1]$ as usual. We have an embedding $h : \text{Rad}(\mathcal{A}) \rightarrow [0.6, 1]$ such that is a homomorphism with respect to $\&$, is monotonic and $h(\bar{1}^{\mathcal{A}}) = 1$. We extend h to $\hat{h} : A \rightarrow [0, 1]$ in the following way:

- $\hat{h}(x) = h(x)$, if $a \in \text{Rad}(\mathcal{A})$,
- $\hat{h}(\neg x) = 1 - h(x)$, if $\neg a \in \neg \text{Rad}(\mathcal{A})$, and
- $\hat{h}(a) = \frac{1}{2}$.

Consider now the algebra \mathcal{B} over $[0.5, 1]$ given by the ordinal sum of the G-chain over $[0.5, 0.6]$ and \mathcal{C} . \mathcal{B} is an MTL-algebra without zero divisors. Consider its connected rotation \mathcal{B}^* defined over $[0, 1]$. Then, $\mathcal{B}^* \in \text{BP}_0^+ \cap \text{IMTL}$ and \hat{h} is an embedding from \mathcal{A} into \mathcal{B}^* , so the theorem holds. \square

Theorem 6.51. *The logic BP_0^{+1} enjoys the SSC.*

Proof: Let $\mathcal{A} \in \text{BP}_0^{+1}$ be a countable perfect chain plus fixpoint. Let $a \in A$ be the fixpoint. The usual method would produce the same problem as in the previous proof, so we will modify it again. If A_+ has minimum or $A_- \setminus \{a\}$ has maximum we embed \mathcal{A} into a new countable perfect chain plus fixpoint in the following way. Let \mathcal{B} be the disconnected rotation of a countable cancellative hoop such that $A \cap B = \emptyset$. We will define a new chain over $C := (A \setminus \{a\}) \cup (B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\})$ by extending the operations and the order of \mathcal{A} and \mathcal{B} in this way:

- $x < y$ for each $x \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\}$ and each $y \in A_+$,
- $x < y$ for each $x \in A_- \setminus \{a\}$ and each $y \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\}$,
- $x \& y = y \& x = y$ for each $x \in A_+$ and each $y \in B \setminus \{\bar{0}^{\mathcal{B}}, \bar{1}^{\mathcal{B}}\}$,

- $x \& y = y \& x = x \&^B y$ for each $x, y \in B \setminus \{\bar{0}^B, \bar{1}^B\}$ such that $x > \neg y$,
- $x \& y = y \& x = \bar{0}^A$ for each $x, y \in B \setminus \{\bar{0}^B, \bar{1}^B\}$ such that $x \leq \neg y$, and
- $x \& y = y \& x = \bar{0}^A$ for each $x, y \in A_- \setminus \{a\}$.

Let \rightarrow be the residuum of $\&$. With this order and these operations \mathcal{C} is a countable perfect MTL-algebra. Then, considering \mathcal{C}^{+1} we obtain a countable perfect chain plus fixpoint where it is possible to embed \mathcal{A} and with no minimum in the set of positives and no maximum in the set of negatives minus the fixpoint. Thus we can suppose without losing generality that \mathcal{A} is such that A_+ has no minimum and $A_- \setminus \{a\}$ has no maximum.

Now we will use the construction of Jenei and Montagna slightly modified. Indeed, we define a densely ordered countable monoid with the lexicographical order and the usual operations, but over the set $\{\langle \bar{0}^A, 1 \rangle, \langle a, 1 \rangle\} \cup \{\langle b, q \rangle : b \in A \setminus \{\bar{0}^A, a\}, q \in Q \cap (0, 1]\}$. To be sure that this also works we only need to check the left-continuity of the monoidal operation on $\langle a, 1 \rangle$. Let $\{\langle b_i, q_i \rangle : i \in \omega\}$ be such that $\sup\{\langle b_i, q_i \rangle : i \in \omega\} = \langle a, 1 \rangle$ and take an arbitrary element $\langle c, p \rangle$. We must prove $\sup\{\langle b_i, q_i \rangle \circ \langle c, p \rangle : i \in \omega\} = \langle a, 1 \rangle \circ \langle c, p \rangle$. If $c \leq a$ then $\langle a, 1 \rangle \circ \langle c, p \rangle = \langle \bar{0}^A, 1 \rangle$ and $\langle b_i, q_i \rangle \circ \langle c, p \rangle = \langle \bar{0}^A, 1 \rangle$ for every $i \in \omega$, so it holds. Suppose that $c > a$. $\langle a, 1 \rangle \circ \langle c, p \rangle = \langle a, 1 \rangle$ and for every $i \in \omega$ we have:

$$\langle b_i, q_i \rangle \circ \langle c, p \rangle := \begin{cases} \langle b_i \& c, q_i \rangle & \text{if } b_i \& c = b_i \\ \langle b_i \& c, 1 \rangle & \text{otherwise.} \end{cases}$$

Now using that $\sup\{b_i \& c : i \in \omega\} = a \& c$ the proof finishes. \square

Finally, we prove that the remaining varieties do not define a logic with standard completeness:

Theorem 6.52. *The logic IBP_0 does not enjoy the SC.*

Proof: This is clear because all IMTL-chains over $[0, 1]$ have negation fixpoint, so there are no perfect standard IMTL-chains. \square

Theorem 6.53. *For every $1 < n < \omega$, the logic BP_0^{+n} does not enjoy the SC.*

Proof: Observe that the only standard chains in BP_0^{+n} are perfect chains plus fixpoint, hence if the standard completeness was true we would have $\text{BP}_0^{+n} = \mathbf{V}(\{\text{standard } \text{BP}_0^{+n}\text{-chains}\}) = \text{BP}_0^{+1}$, a contradiction. \square

6.6 Conclusions

Even though a decomposition theorem in terms of connected rotation-annihilation has not been achieved yet, in this chapter, we have generalized the construction to the general non-involutive case and we have studied some particular cases of this decomposition obtaining several meaningful results.

Table 6.1: Standard completeness properties of fuzzy logics arising from perfect algebras.

Logic	SC	FSSC	SSC
\mathbb{BP}_0	Yes	Yes	Yes
\mathbb{IBP}_0	No	No	No
\mathbb{BP}_0^{+1}	Yes	Yes	Yes
\mathbb{IBP}_0^{+1}	Yes	Yes	Yes
$\mathbb{BP}_0^{+n}, 1 < n < \omega$	No	No	No
$\mathbb{BP}_0^{+\omega}$	Yes	Yes	Yes

- IMTL-chains which are decomposable as a connected rotation-annihilation where the filter is its radical and the convex subalgebra is isomorphic to \mathcal{B}_2 (i.e. are isomorphic to the disconnected rotation of its radical) have been characterized as perfect chains.
- IMTL-chains which are decomposable as a connected rotation-annihilation where the filter is its radical and the convex subalgebra is isomorphic to the drastic product algebra \mathcal{W}_n (i.e. are isomorphic to the disconnected rotation of its radical) have been characterized as perfect chains plus fixpoint and possibly some additional points.
- The varieties generated by perfect IMTL-chains, perfect MTL-chains, perfect IMTL-chains plus fixpoint and perfect MTL-chains plus n points (\mathbb{BP}_0 , \mathbb{IBP}_0 , \mathbb{IBP}_0^{+1} and \mathbb{BP}_0^{+n} respectively) have been finitely axiomatized.
- We have discussed standard completeness properties (see Table 6.1) and Glivenko-style theorems for the logics associated to these new varieties.
- The lattice of subvarieties of \mathbb{IBP}_0 has been proved to be isomorphic to the lattice of varieties of prelinear semihoops, which shows a kind of fractal structure in the lattice of all axiomatic extensions of MTL and suggests its amazing complexity.

Chapter 7

Weakly cancellative MTL-algebras

In Chapter 4 we have proved that every MTL-chain can be decomposed as an ordinal sum of indecomposable totally ordered semihoops. Nevertheless, we also have shown there that the class of all indecomposable totally ordered semihoops seems to be hard to describe, since it contains all IMTL-chains and, as proved in the previous chapter, the complexity of the lattice of varieties of involutive MTL-chains contains at least all the complexity of the lattice of varieties of prelinear semihoops. In this chapter we will study a different class of indecomposable semihoops that seems more accessible than the class of all IMTL-chains. These semihoops are defined by considering a generalization of the property of cancellation that we will call *weak cancellation*.

7.1 The property of weak cancellation

An MTL-chain \mathcal{A} is said to be cancellative if, and only if, for every $a, b, c \in A$ if $a \neq \bar{0}^A$ and $a \& b = a \& c$, then $b = c$. This property is typically satisfied by the product of real numbers. The axiom (II1) was proposed to express the law of cancellation in order to axiomatize the logic of the product t-norm. Nevertheless, (II1) is proved to be equivalent to the property of cancellativity in the presence of the axiom (PC), i.e. it is equivalent to the cancellativity for SMTL-chains. Now we propose an alternative axiom that is equivalent to the cancellativity for all MTL-chains:

$$\neg\psi \vee ((\psi \rightarrow \varphi \& \psi) \rightarrow \varphi) \quad (C)$$

Proposition 7.1. *The variety generated by cancellative MTL-chains is axiomatized by the equation corresponding to axiom (C), i.e. $\neg y \vee ((y \rightarrow x \& y) \rightarrow x) \approx \bar{1}$.*

Proof: Let \mathcal{A} be an MTL-chain. We have to prove that $\mathcal{A} \models \neg y \vee ((y \rightarrow x \& y) \rightarrow x) \approx \bar{1}$ if, and only if, \mathcal{A} is cancellative. First, suppose that the equation is

valid in \mathcal{A} and take $a, b, c \in A$ such that $a \neq \bar{0}^{\mathcal{A}}$ and $a \& b = a \& c$. Then, using the equation we have: $(a \rightarrow b \& a) \rightarrow b = (a \rightarrow c \& a) \rightarrow c = \bar{1}^{\mathcal{A}}$, hence $a \rightarrow b \& a = b$ and $a \rightarrow c \& a = c$, so $b = c$. Conversely, suppose that the chain is cancellative and let us check that for any pair of elements $a, b \in A$ we have $\neg b \vee ((b \rightarrow a \& b) \rightarrow a) = \bar{1}^{\mathcal{A}}$. If $b = \bar{0}^{\mathcal{A}}$, it is obviously true. Otherwise, using the cancellativity we obtain $b \rightarrow a \& b = a$, so the equation is also true. \square

Therefore, in the axiomatization of Product logic and IIMTL we could replace the axiom (II1) by (C). But, in fact, the law of cancellation implies the pseudocomplementation as the following lemma shows.

Lemma 7.2. *Let \mathcal{A} be an MTL-chain. If $\mathcal{A} \models \neg y \vee ((y \rightarrow x \& y) \rightarrow x) \approx \bar{1}$, then $\mathcal{A} \models x \wedge \neg x \approx \bar{0}$.*

Proof: If there exists $a \in A$ such that $a \wedge \neg a \neq \bar{0}^{\mathcal{A}}$, then $a, \neg a \neq \bar{0}^{\mathcal{A}}$. Thus, applying cancellation, from $a \& \neg a = a \& \bar{0}^{\mathcal{A}}$ we obtain $\neg a = \bar{0}^{\mathcal{A}}$, a contradiction. \square

Corollary 7.3. *Π is the axiomatic extension of BL obtained by adding (C) and IIMTL is the axiomatic extension of MTL obtained by adding (C).*

In particular, we have found a new axiomatization for Product logic that is also different from the one proposed¹ by Cintula in [35].

Therefore, cancellativity (C) is a very strong axiom for the axiomatization of Product logic and IIMTL which makes (PC) superfluous. We may wonder if there is an axiom which does not imply (C) but, added to SBL (resp. SMTL) gives an axiomatization of Π (resp. IIMTL). We will prove that the answer to this question is provided by the following weaker form of cancellativity:

Definition 7.4. *Let \mathcal{A} be an MTL-chain. We say that \mathcal{A} is weakly cancellative if, and only if, for every $a, b, c \in A$ if $a \& b = a \& c \neq \bar{0}^{\mathcal{A}}$, then $b = c$.*

Analogously to Proposition 7.1 we can give an equivalent equation for this property:

Proposition 7.5. *Let \mathcal{A} be an MTL-chain. Then, $\mathcal{A} \models \neg(x \& y) \vee ((y \rightarrow x \& y) \rightarrow x) \approx \bar{1}$ if, and only if, \mathcal{A} is weakly cancellative.*

We will refer to the corresponding axiom schema as *axiom of weak cancellation* (WC):

$$\neg(\varphi \& \psi) \vee ((\psi \rightarrow \varphi \& \psi) \rightarrow \varphi) \quad (\text{WC})$$

This axiom turns out to be the difference between pseudocomplementation and cancellation that we were looking for:

¹This axiomatization was also obtained by adding only one axiom with two variables to BL. In fact, it was proved in the same paper that it cannot be done with one axiom in one variable only.

Proposition 7.6. *Let \mathcal{A} be an MTL-chain. Then the following are equivalent:*

- (i) $\mathcal{A} \models x \wedge \neg x \approx \bar{0}$ and $\mathcal{A} \models \neg(x \& y) \vee ((y \rightarrow x \& y) \rightarrow x) \approx \bar{1}$
- (ii) $\mathcal{A} \models \neg y \vee ((y \rightarrow x \& y) \rightarrow x) \approx \bar{1}$

Proof: (ii) \Rightarrow (i): It follows from lemma 7.2. (i) \Rightarrow (ii): Suppose that $a \& b = a \& c$ for some $a, b, c \in A$ with $a \neq \bar{0}^A$. If $a \& b \neq \bar{0}^A$, then by weak cancellation $b = c$. Suppose now that $a \& b = \bar{0}^A$, i.e. $a \leq \neg b$. If $b \neq \bar{0}^A$, then $\neg b = \bar{0}^A$ (by pseudocomplementation), hence $a = \bar{0}^A$, a contradiction. Thus $b = \bar{0}^A$ and analogously $c = \bar{0}^A$, so $b = c$. \square

Another interesting fact about weak cancellation is that (WC) added to IMTL axiomatizes Łukasiewicz logic. Recall that an MTL-algebra satisfying $x \vee y \approx (x \rightarrow y) \rightarrow y$ is already an MV-algebra.

Proposition 7.7. *Let \mathcal{A} be an IMTL-chain. Then, $\mathcal{A} \models x \vee y \approx (x \rightarrow y) \rightarrow y$ if, and only if, $\mathcal{A} \models \neg(x \& y) \vee ((y \rightarrow x \& y) \rightarrow x) \approx \bar{1}$.*

Proof: One direction follows from the fact that all MV-algebras are weakly cancellative. For the other one, suppose that \mathcal{A} is a weakly cancellative IMTL-chain and take a pair of arbitrary elements $a, b \in A$. We have to check that $a \vee b = (a \rightarrow b) \rightarrow b$. If $a \leq b$, it is obvious. Suppose $a > b$, i.e. $\neg b \& a \neq \bar{0}^A$. Then, $(a \rightarrow b) \rightarrow b = \neg b \rightarrow \neg(a \rightarrow b) = \neg b \rightarrow a \& \neg b = a = a \vee b$, by weak cancellation. \square

Corollary 7.8. *Łukasiewicz logic is the axiomatic extension of IMTL obtained by adding the axiom schema (WC).*

Therefore, in the involutive case the property of weak cancellation is not giving any new logic. But in the general case we obtain a new logic and a new variety of MTL-algebras. Let WCMTL be the axiomatic extension of MTL obtained by adding (WC). Of course its equivalent algebraic semantics is the variety of weakly cancellative MTL-algebras, that are called WCMTL-algebras. We will now axiomatize their $\bar{0}$ -free subreducts, the weakly cancellative prelinear semihoops.

Proposition 7.9. *The class of $\bar{0}$ -free subreducts of WCMTL-algebras is the variety of prelinear semihoops satisfying the equation:*

$$(x \& y \rightarrow z) \vee ((y \rightarrow x \& y) \rightarrow x) \approx \bar{1}.$$

Proof: Let \mathcal{A} be a totally ordered semihoop. We have to check that $\mathcal{A} \models (x \& y \rightarrow z) \vee ((y \rightarrow x \& y) \rightarrow x) \approx \bar{1}$ if, and only if, \mathcal{A} is a $\bar{0}$ -free subreduct of some WCMTL-chain \mathcal{C} . First suppose that $\mathcal{A} \models (x \& y \rightarrow z) \vee ((y \rightarrow x \& y) \rightarrow x) \approx \bar{1}$. If there is a minimum element in \mathcal{A} , say m , then we define \mathcal{C} as the \mathcal{L} -expansion of \mathcal{A} where $\bar{0}$ is interpreted as m . It is obvious that \mathcal{C} satisfies the equation of weak cancellation for MTL-algebras. If \mathcal{A} has no minimum, then define $\mathcal{C} := \mathcal{B}_2 \oplus \mathcal{A}$. It is clear that \mathcal{C} is an MTL-chain and \mathcal{A} is one of its $\bar{0}$ -free subreducts. To

check that it satisfies the equation of weak cancellation for MTL-algebras, take an arbitrary pair of elements $a, b \in C$ such that $a \& b \neq \bar{0}^C$ (hence $a, b \in A$). Then, since there is no minimum in \mathcal{A} , there is some $c < a \& b$, hence $a \& b \rightarrow c \neq \bar{1}^A$, which implies $(b \rightarrow a \& b) \rightarrow a = \bar{1}^A$. Conversely, suppose that \mathcal{A} is a $\bar{0}$ -free subreduct of some WCMTL-chain \mathcal{C} . Then for every $a, b, c \in A$, we have $(a \& b \rightarrow c) \vee ((b \rightarrow a \& b) \rightarrow a) \geq (a \& b \rightarrow 0) \vee ((b \rightarrow a \& b) \rightarrow a) = \bar{1}^A$. \square

We will show now that this kind of semihoops gives some examples of indecomposable totally ordered semihoops.

Proposition 7.10. *Let \mathcal{A} be a weakly cancellative totally ordered semihoop. Then:*

- (1) *If \mathcal{A} is unbounded, then it is indecomposable.*
- (2) *Suppose that \mathcal{A} is bounded.*
 - (2.1) *If \mathcal{A} has no zero divisors, then it is a IIMTL-chain and it is decomposable as $\mathcal{A} \cong \mathcal{B}_2 \oplus \mathcal{C}$, where \mathcal{C} is the $\bar{0}$ -free subreduct whose domain is $A \setminus \{\bar{0}^A\}$.*
 - (2.2) *If \mathcal{A} has zero divisors, then it is indecomposable.*

Proof: First suppose that \mathcal{A} is unbounded and decomposable as $\mathcal{A} \cong \mathcal{C}_1 \oplus \mathcal{C}_2$. Then, take $a \in \mathcal{C}_1 \setminus \{\bar{1}^A\}$ and $b \in \mathcal{C}_2 \setminus \{\bar{1}^A\}$. Since it is unbounded there is some $c < a$. Then, the equation of weakly cancellative semihoops would not hold because $a \& b \rightarrow c = a \rightarrow c \neq \bar{1}^A$ and $(a \rightarrow a \& b) \rightarrow b = (a \rightarrow a) \rightarrow b = b \neq \bar{1}^A$. Now suppose that \mathcal{A} is bounded and has no zero divisors. This means that it is pseudocomplemented, hence, by Proposition 7.6 it is cancellative, i.e. a IIMTL-chain. Clearly, it is decomposable as $\mathcal{A} \cong \mathcal{B}_2 \oplus \mathcal{C}$, where $\mathcal{C} = A \setminus \{\bar{1}^A\}$. Suppose that \mathcal{A} is bounded, it has zero divisors and it is decomposable as $\mathcal{A} \cong \mathcal{C}_1 \oplus \mathcal{C}_2$. Then, the existence of zero divisors implies that $\mathcal{C}_1 \not\cong \mathcal{B}_2$. Take $a \in \mathcal{C}_1 \setminus \{\bar{0}^A, \bar{1}^A\}$ and $b \in \mathcal{C}_2 \setminus \{\bar{1}^A\}$. Then, $a \& b \rightarrow \bar{0}^A = a \rightarrow \bar{0}^A \neq \bar{1}^A$ and $(a \rightarrow a \& b) \rightarrow b = (a \rightarrow a) \rightarrow b = b \neq \bar{1}^A$, so \mathcal{A} cannot be weakly cancellative. \square

Given an MTL-chain \mathcal{A} and an element $a \in A$, the truncation of \mathcal{A} with respect to a is the algebra $\mathcal{A}[a] = \langle \{x \in A : a \leq^A x \leq^A \bar{1}^A\}, \&_a^A, \rightarrow_a^A, \leq^A, a, \bar{1}^A \rangle$ where $\&_a^A$ is defined as $x *_{\&_a^A} y = (x *^A y) \vee a$, and \rightarrow_a^A is its residuum (i. e. the restriction of \rightarrow^A to $\{x \in A : a \leq^A x \leq^A \bar{1}^A\}$).

It can be easily checked that any truncation of a IIMTL-chain is a WCMTL-chain. It is well known (see [79]) that each MV-chain is isomorphic to a truncation of some Π -chain, i.e. given an MV-chain \mathcal{A} there is a Π -chain \mathcal{B} and an element $b \in B$ such that $\mathcal{A} \cong \mathcal{B}[b]$. It seems natural to ask whether the same kind of result is true in the general non-divisible case, i.e. whether each WCMTL-chain is isomorphic to a truncation of some IIMTL-chain. We will end the section giving a negative answer to this question by using an example of a totally ordered monoid defined in [57].

For any $a, b, c, d \in \mathbb{N}$, $\langle a, b, c \rangle$ will denote the submonoid of \mathbb{N} generated by a, b, c , and $\langle a, b, c \rangle / d$ will denote the totally ordered monoid obtained by identifying with ∞ all elements of $\langle a, b, c \rangle$ that are greater than or equal to d .

Let $S = \{32^*\} \cup \langle 9, 12, 16 \rangle / 30$ denote the totally ordered monoid obtained from $\langle 9, 12, 16 \rangle / 30$ by adding one additional element, denoted by 32^* . This element satisfies $16 + 16 = 32^*$, $32^* + z = \infty$ for $z \neq 0$, and the whole monoid is to be ordered as follows:

$$0 < 9 < 12 < 16 < 18 < 21 < 24 < 25 < 27 < 28 < 32^* < \infty.$$

All the relations that do not involve 32^* are as in $\langle 9, 12, 16 \rangle / 30$, so we have to only check that $x \leq y$ implies $x + z \leq y + z$ when some of the terms attain the value 32^* . If x or y or z is equal to 32^* then it is easy to see. If $x + z = 32^*$ and $x, z \neq 32^*$ then $x = z = 16$. Thus $32^* = 16 + 16 \leq y + 16$ because if $y > x$ then $y + 16 = \infty$.

Now since we want to make from this monoid an MTL-chain \mathcal{A} , we reverse the order:

$$0 > 9 > 12 > 16 > 18 > 21 > 24 > 25 > 27 > 28 > 32^* > \infty.$$

It is clear that a residuum exists since A is finite and linearly ordered. Even the weak cancellation is satisfied. Suppose that $x + z = y + z \neq \infty$. Then if $x + z = y + z \neq 32^*$ then you can cancel like in \mathbb{N} . If $x + z = y + z = 32^*$ then there are three possibilities: (1): $x = y = z = 16$; (2): $x = 0, z = 32^*$, and $y = 0$; (3): $x = 32^*, z = 0$, and $y = 32^*$. Thus $\mathcal{A} = \langle A, +, \rightarrow, \leq, \infty, 0 \rangle$ is a WCMTL-chain.

Now let us introduce the following identity:

$$(x_1 \& z_1 \rightarrow y_1 \& z_2) \vee (x_2 \& z_2 \rightarrow y_2 \& z_1) \vee (y_1 \& y_2 \rightarrow x_1 \& x_2) \approx \bar{1} \quad (7.1)$$

This identity is not valid in \mathcal{A} . Indeed, let

$$\begin{aligned} x_1 &= 16, & y_1 &= 18, & z_1 &= 16, \\ x_2 &= 12, & y_2 &= 9, & z_2 &= 12. \end{aligned}$$

Then we get the following:

$$\begin{aligned} x_1 + z_1 \rightarrow y_1 + z_2 &= 32^* \rightarrow \infty = 9, \\ x_2 + z_2 \rightarrow y_2 + z_1 &= 24 \rightarrow 25 = 9, \\ y_1 + y_2 \rightarrow x_1 + x_2 &= 27 \rightarrow 28 = 9. \end{aligned}$$

Thus

$$(x_1 + z_1 \rightarrow y_1 + z_2) \vee (x_2 + z_2 \rightarrow y_2 + z_1) \vee (y_1 + y_2 \rightarrow x_1 + x_2) = 9 \neq 0.$$

On the other hand, we claim that given any IIMTL-chain $\mathcal{B} = \langle B, \&, \rightarrow, \leq, 0, \bar{1} \rangle$, every truncation $\mathcal{B}[a] = \langle B[a], \&_a, \rightarrow_a, \leq, a, \bar{1} \rangle$, satisfies the identity (7.1). There are four cases.

1. It is clear that if one of the inequalities $x_1 \&_a z_1 \leq y_1 \&_a z_2$, $x_2 \&_a z_2 \leq y_2 \&_a z_1$, $y_1 \&_a y_2 \leq x_1 \&_a x_2$ is valid then the identity (7.1) is obviously valid.
2. Let y_1 or y_2 equals a . Then $y_1 \&_a y_2 = a \leq x_1 \&_a x_2$.
3. Let z_1 or z_2 equals a . Then either $x_1 \&_a z_1 = a \leq y_1 \&_a z_2$ or $x_2 \&_a z_2 = a \leq y_2 \&_a z_1$.
4. Suppose that $x_1 \&_a z_1 > y_1 \&_a z_2$, $x_2 \&_a z_2 > y_2 \&_a z_1$, and $y_1, y_2, z_1, z_2 > a$. Then we have $x_1 \& x_2 \& z_1 \& z_2 > y_1 \& y_2 \& z_1 \& z_2$ in the original IIMTL-chain \mathcal{B} . By cancellativity of \mathcal{B} we get $x_1 \& x_2 > y_1 \& y_2$ in L . After truncation we obtain that $x_1 \&_a x_2 \geq y_1 \&_a y_2$. Thus the identity (7.1) is valid in this case as well.

Summing up, the identity is valid in all truncations of any IIMTL-chain, but it is not valid in the WCMTL-chain \mathcal{A} . Thus, \mathcal{A} cannot be isomorphic to any truncation of a IIMTL-chain.

7.2 The logics of weakly cancellative chains and their ordinal sums

In the previous section we have defined the logic of weakly cancellative MTL-chains, WCMTL. Now we will consider the logic of ordinal sums of weakly cancellative totally ordered semihoops. This can be done with any axiomatic extension of MTL, so it is worth formulating first this process in an abstract way.

Definition 7.11. *Let L be an axiomatic extension of MTL. We define $\Omega(\mathbb{L})$ as the variety of MTL-algebras generated by all the ordinal sums of $\bar{0}$ -free subreducts of L -chains with the first bounded, and we denote by $\Omega(L)$ its corresponding logic.*

Some well known subvarieties of MTL are closed under this operator, for instance:

- $\Omega(\mathbb{G}) = \mathbb{G}$
- $\Omega(\mathbb{BL}) = \mathbb{BL}$
- $\Omega(\mathbb{SBL}) = \mathbb{SBL}$
- $\Omega(\mathbb{SMTL}) = \mathbb{SMTL}$
- $\Omega(\mathbb{MTL}) = \mathbb{MTL}$

In some other cases they are not closed but we obtain an already known variety:

- $\Omega(\mathbb{BA}) = \mathbb{G}$

- $\Omega(\mathbb{MV}) = \mathbb{BL}$

But sometimes the operator Ω gives new varieties (and hence new fuzzy logics) as we will show now for $\Omega(\text{WCMTL})$ and $\Omega(\text{HMTL})$.

Definition 7.12. Let \mathbb{K} be the variety of MTL-algebras such that letting $x \prec y = x \rightarrow x \& y$ and $I(x) = x \rightarrow x^2$, satisfy the following conditions:

- (1) $(x \wedge y \rightarrow x \& y) \vee I(x \& y) \vee ((x \rightarrow x \& y) \rightarrow y) = 1$
- (2) $(x \prec y) \& (z \rightarrow x) \leq z \prec y$
- (3) $(x \prec y) \& (x \rightarrow z) \& (z \rightarrow y) \leq (z \prec y) \vee (x \prec z) \vee I(x \& y)$

We will prove that $\mathbb{K} = \Omega(\text{WCMTL})$.

Note that $x \prec y = \bar{1}$ if $x \leq y$ and $x \& y = x$. In an ordinal sum of weakly cancellative totally ordered semihoops, this happens if either x is the minimum of the component which y belongs to or $y = \bar{1}$ or $x < y$ and x and y belong to different components. Moreover $I(x) = \bar{1}$ iff x is an idempotent. Thus the intuitive meaning of (1) is that either $x \& y = x$ or $x \& y = y$ or $x \& y$ is an idempotent or x and y belong to the same component and satisfy the cancellation law. The intuitive meaning of (2) is the following: suppose that $x < y$, that x and y are not in the same component and that $z \leq x < y$. Then z and y are in different components. The complementary property is true in all totally ordered semihoops: if $x < y \leq z$ and $x \& y = x$, then $x \& z \geq x \& y = x$, so $x \& z = x$. Finally (3) means that if $x \& y = x$ and x is not an idempotent, then for any z with $x \leq z \leq y$ we must have either $x \& z = x$ or $z \& y = z$.

Lemma 7.13. Equations (1), (2) and (3) hold in any ordinal sum of weakly cancellative totally ordered semihoops whose first component is bounded.

Proof: This is not completely trivial because we have to verify that the equations hold also when the lefthand side is not $\bar{1}$. In the sequel we write $x \ll y$ to mean that $x < y$ and x and y are not in the same component. We also write $x \equiv y$ to mean that x and y are in the same component.

We start from equation (1). The equation clearly holds if $x \not\equiv y$ or if $x \& y$ is an idempotent. If $x \equiv y$ and $x \& y$ is not an idempotent, then $x \& y$ must satisfy the cancellation law and the third disjunct is $\bar{1}$.

Now consider equation (2). The equation clearly holds if $z \prec y = \bar{1}$, hence a fortiori if $z \ll y$. The equation also holds if $z \leq x$ and $x \prec y = \bar{1}$, because then either $z = x$ or $z \ll y$, and in both cases $z \prec y = \bar{1}$. The equation also holds if $x \leq z$, because then $(x \rightarrow x \& y) \& (z \rightarrow x) \leq z \rightarrow z \& y$. It remains to consider the case where $z < x$ and either $y \ll x$ or $x \equiv y$. If $z < x$ and $y \ll x$ then $x \prec y = y$, and (2) becomes $y \leq z \prec y$, which is clearly satisfied. Finally suppose $z < x$ and $x \equiv y$. Without loss of generality we can also suppose $z \equiv x \equiv y$, otherwise $z \ll y$ and $z \prec y = \bar{1}$. Thus (2) becomes $x \prec y \leq z \prec y$. If $z \& y$ is not an idempotent, then $x \prec y = z \prec y = y$ and (2) holds. If $x \& y$ is an idempotent, then $x \& y = z \& y$ is the minimum m of the component which x, y, z belong to, and (2) becomes $x \rightarrow m \leq z \rightarrow m$, which clearly holds as $z < x$. Finally if $z \& y = m$ is an idempotent and $x \& y$ is not (so $x \& y > m$), then $x \prec y = y$ and

$z \prec y = z \rightarrow m$. Now from $z \& y = m$ by residuation we derive $y \leq z \rightarrow m$ and the claim is proved.

We verify (3). Note that (3) holds (in any ordinal sum of WCMTL semihoops) if either $x \& y$ is an idempotent or $z \prec y = \bar{1}$ or $x \prec z = \bar{1}$ (thus in particular if $z \ll y$ or $x \ll z$). Thus we suppose that none of the above conditions holds. If $y \ll z$ then $z \rightarrow y = y$ and (3) holds. If $z \ll x$ then $x \rightarrow z = z$ and (3) holds. It remains to consider the case where $x \equiv z \equiv y$. In this case, since we have excluded that $x \& y$ is an idempotent, we have $x \prec y = y$. Now let C be the component which x, y, z belong to. If either C has no minimum or $z \& y$ is not the minimum of C , then $x \prec y = z \prec y = y$, and (3) is verified. If C has a minimum m and $z \& y = m$, then $x \prec y = y \leq z \rightarrow m = z \prec y$ and once again (3) is verified. \square

Lemma 7.14. *Let \mathcal{A} be an MTL-chain which satisfies (1), (2) and (3). Then \mathcal{A} is the ordinal sum of an ordered family of weakly cancellative totally ordered semihoops, whose first component is bounded.*

Proof: By Theorem 4.54 any linearly ordered MTL-algebra can be decomposed as an ordinal sum of sum-indecomposable totally ordered semihoops, with the first bounded. So it is sufficient to prove that a sum-indecomposable linearly ordered semihoop satisfying (1), (2) and (3) is weakly cancellative. Let \mathcal{C} be such a semihoop. We claim that \mathcal{C} has no idempotent elements except from its maximum and its minimum (if such a minimum exists). Suppose by contradiction that u is idempotent and that there are $a, b \in \mathcal{C}$ with $a < u < b$. Then $x \& u = u$ for all $x \geq u$, and by (2), for all $z \leq u \leq v$ one has $z \& v = z$. Then $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$ where $\mathcal{C}_2 = \{z : z \geq u\}$ and $\mathcal{C}_1 = (\mathcal{C} \setminus \mathcal{C}_2) \cup \{\bar{1}^{\mathcal{C}}\}$, contradicting our assumption that \mathcal{C} is sum-indecomposable. We now prove that if both x and y are not idempotent, then $x \& y < x \wedge y$. The claim is obvious if $x = y$ so we can assume without loss of generality that $x < y$. The claim is also obvious if $x \& y$ is the minimum m of \mathcal{C} , because m is an idempotent and x, y are not such, so $m = x \& y < x \wedge y$. Thus suppose by contradiction that there is $z \in \mathcal{C}$ such that $z < x \& y = x \wedge y = x < y$. Since $x \& y$ is not an idempotent, by axiom (3), for any u with $x \leq u \leq y$ we have either $x \& u = x$ or $u \& y = u$. Now let $\mathcal{C}_1 = \{u : u \& y = u\} \cup \{\bar{1}^{\mathcal{C}}\}$ and $\mathcal{C}_2 = (\mathcal{C} \setminus \mathcal{C}_1) \cup \{1\}$. $\mathcal{C}_1 \setminus \{\bar{1}^{\mathcal{C}}\}$ is downwards closed, so for all $w \in \mathcal{C}_2$ and for all $z \in \mathcal{C}_1 \setminus \{\bar{1}^{\mathcal{C}}\}$ we have $z \leq w$. We claim that for all $w \in \mathcal{C}_2$ and for all $z \in \mathcal{C}_1 \setminus \{\bar{1}^{\mathcal{C}}\}$ we have $z \& w = z$. This implies that $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$, which is impossible. Thus let $w \in \mathcal{C}_2$ and $z \in \mathcal{C}_1 \setminus \{\bar{1}^{\mathcal{C}}\}$. We can assume without loss of generality that z is not an idempotent, otherwise z is the minimum of \mathcal{C} and the claim is trivial. Moreover by the definition of \mathcal{C}_1 we have that $z \& y = z$. So if $w \geq y$, we have $z \& w = z$ as desired. If $w < y$, then since $z \& y = z$ is not an idempotent, by axiom (3) with x replaced by z we have that either $w \& y = w$ or $z \& w = z$. But $w \& y = w$ is excluded, because $w \in \mathcal{C}_2$. So $z \& w = z$ and the proof is complete. \square

Thus we obtain a finite axiomatization for the variety generated by those ordinal sums:

Theorem 7.15. \mathbb{K} is the variety generated by the ordinal sums of weakly cancellative totally ordered semihoops (with the first bounded), i.e. $\mathbb{K} = \Omega(\text{WCMTL})$.

Now consider the variety $\Omega(\Pi\text{MTL})$. Adapting slightly the axiomatization and the proof of the last theorem we obtain the following result.

Theorem 7.16. The variety $\Omega(\Pi\text{MTL})$ generated by ordinal sums of cancellative semihoops (with the first bounded) is axiomatized by:

- (1') $(x \wedge y \rightarrow x \& y) \vee I(x) \vee ((x \rightarrow x \& y) \rightarrow y) \approx \bar{1}$
- (2) $(x \prec y) \& (z \rightarrow x) \leq z \prec y$
- (3) $(x \prec y) \& (x \rightarrow z) \& (z \rightarrow y) \leq (z \prec y) \vee (x \prec z) \vee I(x \& y)$

Accordingly, we define the corresponding logics. The logic $\Omega(\text{WCMTL})$ is the axiomatic extension of MTL obtained by adding the following schemata:

- (a) $(\varphi \wedge \psi \rightarrow \varphi \& \psi) \vee I(\varphi \& \psi) \vee ((\varphi \rightarrow \varphi \& \psi) \rightarrow \psi)$
- (b) $(\varphi \prec \psi) \& (\chi \rightarrow \varphi) \rightarrow \chi \prec \psi$
- (c) $(\varphi \prec \psi) \& (\varphi \rightarrow \chi) \& (\chi \rightarrow \psi) \rightarrow (\chi \prec \psi) \vee (\varphi \prec \chi) \vee I(\varphi \& \psi)$

and the logic $\Omega(\Pi\text{MTL})$ is the axiomatic extension of MTL obtained by adding the following schemata:

- (a') $(\varphi \wedge \psi \rightarrow \varphi \& \psi) \vee I(\varphi) \vee ((\varphi \rightarrow \varphi \& \psi) \rightarrow \psi)$
- (b) $(\varphi \prec \psi) \& (\chi \rightarrow \varphi) \rightarrow \chi \prec \psi$
- (c) $(\varphi \prec \psi) \& (\varphi \rightarrow \chi) \& (\chi \rightarrow \psi) \rightarrow (\chi \prec \psi) \vee (\varphi \prec \chi) \vee I(\varphi \& \psi)$

Let (OS) be the conjunction of the schemata (a), (b) and (c), and let (OS') be the conjunction of the schemata (a'), (b) and (c). Adding combinations of the schemata (WC), (PC), (OS), (OS'), (Div) and (Inv) to MTL we obtain the hierarchy of logics depicted in figure 7.1, where CPC is the Classical Propositional Calculus and the following two new logics appear:

- $\text{S}\Omega(\text{WCMTL})$ is $\Omega(\text{WCMTL})$ plus (PC).
- WCBL is BL plus (WC).

7.3 LF, FEP and FMP in weakly cancellative fuzzy logics

We will study some properties of these logics and their corresponding varieties of MTL-algebras.

Lemma 7.17. *Let \mathcal{A} be an MTL-chain. Then, \mathcal{A} is a WCBL-chain if, and only if, it is an MV-chain or a Π -chain.*

Proof: One direction is trivial. For the other one, let \mathcal{A} be a WCBL-chain and consider its decomposition as ordinal sum of Wajsberg hoops (with the first bounded), $\mathcal{A} \cong \bigoplus_{i \in I} \mathcal{C}_i$. If $|I| = 1$, then $\mathcal{A} \cong \mathcal{C}_{i_0}$ is an MV-chain. If $|I| > 1$ it must be of the form $\mathcal{A} \cong \mathcal{B}_2 \oplus \mathcal{C}$, with \mathcal{C} cancellative (otherwise the weak cancellation would not be satisfied), hence it is a Π -chain. \square

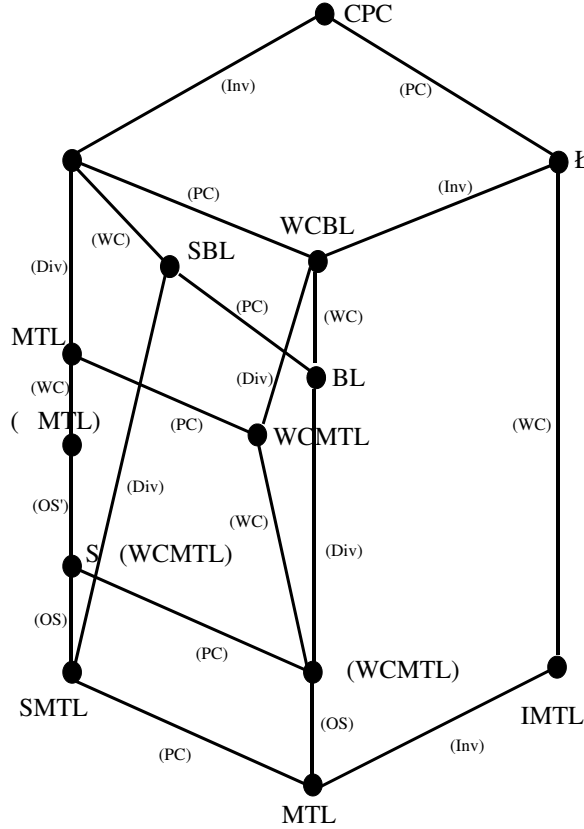


Figure 7.1: Graphic of axiomatic extensions of MTL obtained by adding combinations of the schemata (WC), (PC), (OS), (Div) and (Inv). All the depicted inclusions are proper.

Proposition 7.18. *WCBL is the infimum of Π and \mathbb{L} in the lattice of axiomatic extensions of MTL. Thus, $\text{WCBL} = \mathbf{V}([0, 1]_{\mathbb{L}}, [0, 1]_{\Pi})$ and WCBL enjoys the FSSC.*

Proof: It follows directly from the previous lemma. \square

Therefore, WCBL is the logic LII defined in [30] for which we have found now an alternative axiomatization.

Corollary 7.19. *WCBL does not have the finite model property.*

Proof: Suppose WCBL has the FMP. Then, WCBL would be generated as a variety by the finite WCBL-chains, but since there are no finite Π -chains with more than two elements, it would be generated by finite MV-algebras, so $\text{WCBL} = \text{MV}$, a contradiction. \square

Thus WCBL lacks also the FEP. Nevertheless, WCBL logic is still decidable, since it is the infimum of Π and L and those logics are decidable.

Proposition 7.20. *Let \mathcal{A} be an MTL-chain. Then, \mathcal{A} is an $\text{S}\Omega(\text{WCMTL})$ -chain if, and only if, it is an ordinal sum of totally ordered weakly cancellative semihoops such that the first one is a IIMTL-chain.*

Proof: One direction is trivial. For the other one, let \mathcal{A} be an $\text{S}\Omega(\text{WCMTL})$ -chain. In particular it is an $\Omega(\text{WCMTL})$ -chain, so it is decomposable as an ordinal sum of totally ordered weakly cancellative semihoops with the first one bounded. Then, it is obvious that the axiom (PC) implies that the first component must be an SMTL-chain, hence a IIMTL-chain. \square

Finally, we will prove that the FMP fails for all logics between $\Omega(\text{WCMTL})$ and IIMTL (both included). First we need some lemmata.

Lemma 7.21. *Each finite WCMTL-chain \mathcal{A} is Archimedean, i.e. for any $\bar{0}^{\mathcal{A}} < x < y < 1$ there is n such that $y^n \leq x$.*

Proof: Suppose not. Then $x < y^n$ for all n . Since $y^n \neq \bar{0}^{\mathcal{A}}$ for all n , we have $y > y^2 > y^3 > \dots$ by weak cancellativity. Thus \mathcal{A} must be infinite, a contradiction. \square

Lemma 7.22. *Let \mathcal{A} be an MTL-chain and $p, q \in A$. If $p \rightarrow q = q$ then $q = \max [q]_{F(p)}$.*

Proof: Assume that $p \rightarrow q = q$. Suppose that $z \in [q]_{F(p)}$. Then $z \rightarrow q \in F(p)$. Thus there exists $n \in \omega$ such that $p^n \leq z \rightarrow q$. By residuation we get $z \leq p^n \rightarrow q$. Since we assume that $p \rightarrow q = q$, we have $p^n \rightarrow q = p^{n-1} \rightarrow (p \rightarrow q) = p^{n-1} \rightarrow q = q$. Thus we obtain that $z \leq q$. Hence $q = \max [q]_{F(p)}$. \square

Lemma 7.23. *Let \mathcal{A} be an Archimedean MTL-chain. Then \mathcal{A} is either a BL-chain or it has a co-atom.*

Proof: Suppose that there is no co-atom. Then we will show that the divisibility condition, $a \wedge b = a \& (a \rightarrow b)$, holds in \mathcal{A} . If $a \leq b$ or a equals $\bar{1}^{\mathcal{A}}$, then the equality trivially holds. If $a \rightarrow b = \bar{0}^{\mathcal{A}}$ then $b = \bar{0}^{\mathcal{A}}$ and the equality again holds. Thus suppose that $a > b$, $a, b \neq \bar{1}^{\mathcal{A}}$, and $a \rightarrow b > \bar{0}^{\mathcal{A}}$. By residuation we get $a \& (a \rightarrow b) \leq b$. Suppose that $a \& (a \rightarrow b) < b$. Let $M = A \setminus \{\bar{1}^{\mathcal{A}}\}$. Clearly $\bigvee M = \bar{1}^{\mathcal{A}}$ because there is no co-atom. Since \mathcal{A} is Archimedean, we get that for each $r \in M$ there exists $k_r \in \omega$ (possibly 0) such that

$$r^{k_r+1} \leq a \rightarrow b < r^{k_r}.$$

Thus we obtain for all $r \in M$:

$$a \& r^{k_r+1} \leq a \& (a \rightarrow b) < b < a \& r^{k_r}.$$

The last inequality holds since $a \rightarrow b$ is the maximal solution of the inequality $a \& x \leq b$ and $a \rightarrow b < r^{k_r}$.

Further, from the existence of residuum we get $\bigvee_{r \in M} (b \& r) = b \& \bigvee M = b$. Hence there must be an $s \in M$ such that $a \& (a \rightarrow b) < b \& s$. Thus we obtain

$$a \& s^{k_s+1} \leq a \& (a \rightarrow b) < b \& s \leq a \& s^{k_s+1},$$

a contradiction. □

Lemma 7.24. *In each Archimedean WCMTL-chain \mathcal{A} the identity*

$$((p \rightarrow q) \rightarrow q)^2 \leq p \vee q \vee \neg q \tag{7.2}$$

is valid.

Proof: If there is no co-atom, then by Lemma 7.23, \mathcal{A} is a WCBL-chain hence either a Π -chain or an MV-chain. But in any Π -chain or MV-chain the identity (7.2) is valid.

Thus suppose that there is a co-atom a . The only interesting case is for $\bar{1}^{\mathcal{A}} > p > q > \bar{0}^{\mathcal{A}}$. We can also assume that $p \rightarrow q > \bar{0}^{\mathcal{A}}$ otherwise $q = \bar{0}^{\mathcal{A}}$. Since \mathcal{A} is Archimedean, there is $n \in \omega$ such that

$$a^{n+1} \leq p \rightarrow q < a^n.$$

Since $a^n > p \rightarrow q$, we get $a^n \rightarrow q < p$ (if $p \leq a^n \rightarrow q$ then $a^n \leq p \rightarrow q$). It follows that

$$(p \rightarrow q) \rightarrow q \leq a^{n+1} \rightarrow q = a \rightarrow (a^n \rightarrow q) \leq a \rightarrow p.$$

Thus $(p \rightarrow q) \rightarrow q \leq a \rightarrow p$.

Now we claim that $(p \rightarrow q) \rightarrow q \leq a$. If not then $(p \rightarrow q) \rightarrow q = \bar{1}^{\mathcal{A}}$, i.e. $p \rightarrow q = q$. Thus by Lemma 7.22 we have $q = \max [q]_{F(p)}$. Since \mathcal{A} is Archimedean, $F(p)$ equals either to A or to $A \setminus \{\bar{0}^{\mathcal{A}}\}$. Thus $q \in F(p)$ and $q = \bar{1}^{\mathcal{A}}$. But we assume that $\bar{1}^{\mathcal{A}} > p > q > \bar{0}^{\mathcal{A}}$. Hence $(p \rightarrow q) \rightarrow q \leq a$.

Finally, we get

$$((p \rightarrow q) \rightarrow q)^2 \leq a \& ((p \rightarrow q) \rightarrow q) \leq a \& (a \rightarrow p) \leq p \leq p \vee q \vee \neg q.$$

□

Let $\varphi = (q \rightarrow (p \& q)) \rightarrow p$, $\psi = (p \rightarrow q) \rightarrow q$, and $\chi = p \vee q \vee \neg q$.

Lemma 7.25. *In any finite $\Omega(\text{WCMTL})$ -chain \mathcal{A} the identity $\varphi \wedge \psi^2 \leq \chi$ is valid.*

Proof: If $p \leq q$ then $\psi = q$ and $\varphi \wedge \psi^2 = \varphi \wedge q^2 \leq \chi$. Thus let us suppose that $p > q$.

First, let p, q belong to different components. Then $\varphi = (q \rightarrow q) \rightarrow p = p$. Thus $\varphi \wedge \psi^2 \leq \varphi = p \leq \chi$.

Second, let p, q be in the same component. This component is a $\bar{0}$ -free subreduct of a finite WCMTL-chain \mathcal{W} . By Lemma 7.21 we know that \mathcal{W} is Archimedean. Thus by Lemma 7.24 we get that $\psi^2 \leq \chi$ is valid in \mathcal{W} . Since \mathcal{W} is a subalgebra of \mathcal{A} , we get that $\varphi \wedge \psi^2 \leq \psi^2 \leq \chi$ is valid in \mathcal{A} . □

Lemma 7.26. *There is a IIMTL-chain \mathcal{A} such that $\varphi \wedge \psi^2 \leq \chi$ is not valid in \mathcal{A} .*

Proof: Consider the algebra \mathcal{A} defined as follows:

- The domain of \mathcal{A} is $\{\langle 0, 0 \rangle\} \cup ((0, 1] \times (0, 1])$.
- The lexicographic order \leq_{lex} defines the lattice structure.
- Multiplication is defined componentwise.
- Implication \Rightarrow is defined as follows: if $\langle a, b \rangle \leq_{lex} \langle c, d \rangle$, then $\langle a, b \rangle \Rightarrow \langle c, d \rangle = \langle 1, 1 \rangle$; if $\langle a, b \rangle \neq \langle 0, 0 \rangle$, then $\langle a, b \rangle \Rightarrow \langle 0, 0 \rangle = \langle 0, 0 \rangle$; if $a, b, c, d > 0$ and $a \geq c$ and $b \geq d$, then $\langle a, b \rangle \Rightarrow \langle c, d \rangle = \langle \frac{c}{a}, \frac{d}{b} \rangle$; if $a > c$ and $b \leq d$, then $\langle a, b \rangle \Rightarrow \langle c, d \rangle = \langle \frac{c}{a}, 1 \rangle$.

It is readily seen that \mathcal{A} is a IIMTL-algebra. For $e(p) = \langle 1, \frac{1}{2} \rangle$ and $e(q) = \langle \frac{1}{2}, 1 \rangle$, we have $e(\varphi) = e(\psi^2) = \langle 1, 1 \rangle$ and $e(\chi) \neq \langle 1, 1 \rangle$. □

Thus we get the following theorem.

Theorem 7.27. *If \mathbb{K} is a variety such that $\text{IIMTL} \subseteq \mathbb{K} \subseteq \Omega(\text{WCMTL})$, then \mathbb{K} has not the FMP (and hence also the FEP is false in \mathbb{K}).*

Proof: Let \mathcal{A} be the chain defined in the previous lemma. Therefore, \mathcal{A} is an infinite chain of \mathbb{K} where $\varphi \wedge \psi^2 \leq \chi$ fails, but by Lemma 7.25, the equation is valid in all the finite chains of \mathbb{K} . □

7.4 On standard completeness theorems

In this section we discuss the standard completeness of the logics introduced so far and of their first-order extensions.

Theorem 7.28. *WCMTL enjoys the FSSC.*

Proof: We will prove it by following the method used in [88] and its modification from [89] for the FSSC of IIMTL, so we will not check again the details that are already done there. Take a finite set $T \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ such that $T \not\models_{WCMTL} \varphi$. Then, there is a WCMTL-chain $\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ and an evaluation $e : Fm_{\mathcal{L}} \rightarrow \mathcal{A}$ such that $e[T] \subseteq \{\bar{1}^{\mathcal{A}}\}$ and $e(\varphi) \neq \bar{1}^{\mathcal{A}}$. Consider the set $G := \{e(\psi) : \psi \text{ is a subformula of some formula of } T \cup \{\varphi\}\}$. G is finite because T is. Let \mathcal{S} be the submonoid of \mathcal{A} generated by G . As in [88], \mathcal{S} is residuated and the residuum is given by: $a \rightarrow b = \max\{z \in \mathcal{S} : a \& z \leq b\}$. Thus, the enriched submonoid $\mathcal{S} = \langle S, \&, \rightarrow, \wedge, \vee, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ is a countable MTL-chain. Moreover, since its monoidal operation is just the restriction of the monoidal operation of \mathcal{A} , it is clear that it is also weakly cancellative, hence a WCMTL-chain. Now we consider the evaluation $e' : Fm_{\mathcal{L}} \rightarrow \mathcal{S}$ such that for every propositional variable v ,

$$e'(v) = \begin{cases} e(v) & \text{if } v \text{ appears in some formula of } T \cup \{\varphi\} \\ \bar{0}^{\mathcal{A}} & \text{otherwise.} \end{cases}$$

One can prove by induction that $e'(\psi) = e(\psi)$ for every ψ a subformula of some formula of $T \cup \{\varphi\}$. Furthermore, since \mathcal{S} is generated from a finite set by using the monoidal operation, then it has only a finite number of Archimedean classes.

Now define a new chain over the set $S' := \{\langle s, r \rangle : s \in S \setminus \{\bar{0}^{\mathcal{A}}\}, r \in (0, 1]\} \cup \{\langle \bar{0}^{\mathcal{A}}, 1 \rangle\}$, with the lexicographical order \leq_{lex} and the following operations:

$$\begin{aligned} \langle a, x \rangle \&' \langle b, y \rangle &= \begin{cases} \langle \bar{0}^{\mathcal{A}}, 1 \rangle & \text{if } a \& b = \bar{0}^{\mathcal{A}}, \\ \langle a \& b, xy \rangle & \text{otherwise.} \end{cases} \\ \langle a, x \rangle \rightarrow' \langle b, y \rangle &= \begin{cases} \langle a \rightarrow b, 1 \rangle & \text{if } a \& (a \rightarrow b) < b, \\ \langle a \rightarrow b, \min\{1, y/x\} \rangle & \text{otherwise.} \end{cases} \end{aligned}$$

$\mathcal{S}' = \langle S', \&', \rightarrow', \leq_{lex}, \langle \bar{0}^{\mathcal{A}}, 1 \rangle, \langle 1^{\mathcal{A}}, 1 \rangle \rangle$ is an MTL-chain with a finite number of Archimedean classes, and there is an embedding $\Psi : \mathcal{S} \rightarrow \mathcal{S}'$ defined by $\Psi(a) = \langle a, 1 \rangle$. Moreover \mathcal{S}' is weakly cancellative. Indeed, if $\langle a, x \rangle, \langle b, y \rangle, \langle c, z \rangle \in \mathcal{S}'$ are such that $\langle a, x \rangle \&' \langle b, y \rangle = \langle a, x \rangle \&' \langle c, z \rangle \neq \langle \bar{0}^{\mathcal{A}}, 1 \rangle$, then $\langle a \& b, xy \rangle = \langle a \& c, xz \rangle \neq \langle \bar{0}^{\mathcal{A}}, 1 \rangle$. Thus, $a \& b = a \& c \neq \bar{0}^{\mathcal{A}}$ and $xy = xz \neq 0$ which, using the weak cancellation of \mathcal{A} and the cancellation of the product of reals, implies $b = c$ and $y = z$.

Finally, as in [89] the set S' is order isomorphic to the real unit interval $[0, 1]$, so there is a standard WCMTL-chain \mathcal{B} and an isomorphism $\Phi : \mathcal{S}' \rightarrow \mathcal{B}$. This

standard chain and the evaluation $\Phi \circ \Psi \circ e'$ are a countermodel for the derivation of φ from T . \square

Theorem 7.29. $\Omega(\text{WCMTL})$, $\text{S}\Omega(\text{WCMTL})$ and $\Omega(\Pi\text{MTL})$ enjoy the FSSC.

Proof: Consider first the $\Omega(\text{WCMTL})$ case. The first part of the proof runs parallel to the previous one. Take a finite set $T \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$ such that $T \not\vdash_{\Omega(\text{WCMTL})} \varphi$. Then, there is a $\Omega(\text{WCMTL})$ -chain $\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ and an evaluation $e : \text{Fm}_{\mathcal{L}} \rightarrow \mathcal{A}$ such that $e[T] \subseteq \{\bar{1}^{\mathcal{A}}\}$ and $e(\varphi) \neq \bar{1}^{\mathcal{A}}$. Consider the set $G := \{e(\psi) : \psi \text{ is a subformula of some formula of } T \cup \{\varphi\}\}$. G is finite because T is. Let \mathcal{S} be the submonoid of \mathcal{A} generated by G . Again it is residuated, so we have an enriched submonoid $\mathcal{S} = \langle S, \&, \rightarrow, \wedge, \vee, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ such that is a countable MTL-chain (with a finite number of Archimedean classes). Moreover, since its monoidal operation is just the restriction of the monoidal operation of \mathcal{A} , it is clear that it is also an ordinal sum of weakly cancellative totally ordered semihoops with the first bounded, hence a $\Omega(\text{WCMTL})$ -chain. Since it is finitely generated, this ordinal sum must have a finite number of components, say $\mathcal{S} = \bigoplus_{i < k} \mathcal{C}_i$ for some natural number k . Now we consider the evaluation $e' : \text{Fm}_{\mathcal{L}} \rightarrow \mathcal{S}$ such that for every propositional variable v ,

$$e'(v) = \begin{cases} e(v) & \text{if } v \text{ appears in some formula of } T \cup \{\varphi\} \\ \bar{0}^{\mathcal{A}} & \text{otherwise.} \end{cases}$$

Again, by induction, it is provable that $e'(\psi) = e(\psi)$ for every ψ a subformula of some formula of $T \cup \{\varphi\}$.

Finally, applying to every weakly cancellative totally ordered semihoop of the ordinal sum the construction of the proof of the previous theorem, we have for every $i < k$ an embedding $\mathcal{C}_i \hookrightarrow [0, 1]_{\&_i}$ into a standard WCMTL-chain. Therefore, there is an embedding $f : \mathcal{S} \hookrightarrow \bigoplus_{i < k} [0, 1]_{\&_i}$. It is clear that $\bigoplus_{i < k} [0, 1]_{\&_i}$ is isomorphic to a standard $\Omega(\text{WCMTL})$ -chain. This standard $\Omega(\text{WCMTL})$ -chain with the evaluation $f \circ e'$ gives the desired countermodel for the derivation of φ from T .

For the cases of $\text{S}\Omega(\text{WCMTL})$ and $\Omega(\Pi\text{MTL})$ the proof is similar. For the first one we only need to realize that the first component of the ordinal sum of \mathcal{S} now is a ΠMTL -chain and it will be embedded into a standard ΠMTL -chain, so in the end we will get a standard $\text{S}\Omega(\text{WCMTL})$ -chain. For the second one, notice that all the components of \mathcal{S} are cancellative so they embed into standard ΠMTL -chains, so in the end a standard $\Omega(\Pi\text{MTL})$ -chain is obtained. \square

Furthermore, taking into account that in the proofs of the last two theorems the standard chains that are built have only finitely many Archimedean classes, we can improve the finite standard completeness results by considering only the semantics given by standard chains with a finite number of Archimedean classes.

Corollary 7.30. *If L is a logic from the set $\{\text{WCMTL}, \Omega(\text{WCMTL}), \Omega(\Pi\text{MTL}), \text{S}\Omega(\text{WCMTL})\}$, then for every finite set of formulae $T \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$*

we have:

$T \vdash_L \varphi$ if, and only if, $T \models_{\mathcal{A}} \varphi$ for every standard L -chain \mathcal{A} with finitely many Archimedean classes.

Now we will prove that no logic between $\Omega(\text{WCMTL})$ and Π (both included) enjoys the SSC. Consider the following set Γ of sentences in a language whose propositional variables are $p_0, \dots, p_n, \dots, p_\omega$:

1. $p_i \leftrightarrow p_{i+1}^2 \ (i \in \omega)$.
2. $\neg\neg p_0$.
3. $p_i \rightarrow p_\omega \ (i \in \omega)$.

Claim (A). For any standard $\Omega(\text{WCMTL})$ -algebra \mathcal{A} one has:

$\Gamma \models_{\mathcal{A}} p_0 \rightarrow p_\omega \& p_0$

Proof of Claim (A). Suppose that all formulas of Γ are satisfied in \mathcal{A} under some evaluation e . Let, for $k = 0, 1, \dots, n, \dots, \omega$, $a_k = e(p_k)$. Then by (2), $a_0 \neq 0$ and by (1) and (3), for all $k \in \omega$ we have $a_{k+1}^2 = a_k$ and $a_k \leq a_\omega$. So all a_i with $i < \omega$ are in the same component.

If $a_\omega = 1$ the result is obvious. Suppose $a_\omega < 1$. Let $a = \sup \{a_k : k \in \omega\}$ (such a supremum exists by the completeness of $[0, 1]$). Then $a \leq a_\omega$. Moreover by the left-continuity of the monoidal operation \cdot , we have $a^2 = \sup \{a_{k+1}^2 : k \in \omega\} = \sup \{a_k : k \in \omega\} = a$. So a is an idempotent, between a_0 and a_ω . It follows that a_ω and a_0 are in different components, therefore $a_0 \cdot a_\omega = a_0$, and the claim is proved.

Claim (B). There are a product algebra \mathcal{B} and an evaluation e in \mathcal{B} such that $e(A) = 1$ for all $A \in \Gamma$ and $e(p_\omega \cdot p_0) < e(p_0)$.

Proof of Claim (B). Let $\mathcal{B} = \{\langle 0, 0 \rangle\} \cup \{\langle 1, p \rangle : 0 < p \leq 1\} \cup ((0, 1) \times (0, +\infty))$, ordered by the lexicographic order \leq_{lex} (thus if $0 < a < b \leq 1$ then for any $c, d \in (0, +\infty)$, one has $\langle a, c \rangle <_{lex} \langle b, d \rangle$) and having ordinary product (defined componentwise) as monoidal operation. Thus our algebra consists of $\langle 0, 0 \rangle$ plus the negative cone of the multiplicative group $(0, +\infty)^2$ ordered lexicographically. Here the identity is $\langle 1, 1 \rangle$, therefore *negative* means *less than* $\langle 1, 1 \rangle$. In other words, it is a product algebra. Now define inductively $e(p_0) = \langle \frac{1}{2}, 1 \rangle$, $e(p_{i+1}) = \langle \sqrt{e(p_i)}, 1 \rangle$. Further, define $e(p_\omega) = \langle 1, \frac{1}{2} \rangle$. It is immediate to verify that $e(A) = \langle 1, 1 \rangle$ for any $A \in \Gamma$ and that $e(p_\omega \cdot p_0) = \langle \frac{1}{2}, \frac{1}{2} \rangle <_{lex} \langle \frac{1}{2}, 1 \rangle = e(p_0)$. This concludes the proof of Claim (B).

Theorem 7.31. *No propositional logic between $\Omega(\text{WCMTL})$ and Product logic Π (both included) enjoys the SSC.*

Proof: Let L be such a logic and \mathbb{L} the corresponding variety. Then the standard elements of \mathbb{L} are standard $\Omega(\text{WCMTL})$ -algebras and the algebra \mathcal{B} in Claim B is in \mathbb{L} . Hence $\Gamma \models_{\mathcal{A}} p_0 \rightarrow p_\omega \& p_0$ holds in any standard algebra \mathcal{A} in \mathbb{L} , but not in all algebras in \mathbb{L} (\mathcal{B} is a counterexample). It follows that $\Gamma \not\vdash_L p_0 \rightarrow p_\omega \& p_0$. \square

Table 7.1: Logical and algebraic properties of weakly cancellative fuzzy logics

Logic	LF	FEP	FMP	Decidable	FSSC	SSC
$\Omega(\text{WCMTL})$	No	No	No	?	Yes	No
$\text{S}\Omega(\text{WCMTL})$	No	No	No	?	Yes	No
WCMTL	No	No	No	?	Yes	No
$\Omega(\text{IIMTL})$	No	No	No	?	Yes	No
IIMTL	No	No	No	Yes	Yes	No
WCBL	No	No	No	Yes	Yes	No
Π	No	No	No	Yes	Yes	No
\mathbb{L}	No	Yes	Yes	Yes	Yes	No

Corollary 7.32. *The following logics do not enjoy the SSC: Π , WCBL, SBL, BL, IIMTL, $\Omega(\text{IIMTL})$, WCMTL, $\text{S}\Omega(\text{WCMTL})$ and $\Omega(\text{WCMTL})$.*

7.5 Conclusions

We have introduced and studied the property of weak cancellation and we have obtained the following results:

- In Chapter 4 we have proved a theorem of representation of MTL-chains as ordinal sums of indecomposable totally ordered semihoops. A characterization of such indecomposable semihoops is still not known, but weak cancellation gives a big and interesting class of indecomposable totally ordered semihoops.
- Weak cancellation gives a new way to define Łukasiewicz logic from IMTL.
- Weak cancellation is exactly the difference between cancellation and pseudocomplementation, so it gives an alternative axiomatization of Π and IIMTL and allows to define a new hierarchy of fuzzy logics.
- The ordinal sums of weakly cancellative totally ordered semihoops define a new logic, $\Omega(\text{WCMTL})$, that it is analogous to BL, in the sense that all BL-chains are decomposable as ordinal sums of Wajsberg hoops (hence weakly cancellative).
- We have studied some properties of these weakly cancellative fuzzy logics, but the decidability problem remains open in general. It has been proved true very recently for IIMTL in [90], but we do not whether this proof can be extended to other weakly cancellative logics. The studied properties are gathered in the Table 7.1.

Chapter 8

n -contractive MTL-algebras

8.1 The n -contraction

In [104] (it is an unpublished work, but all its results can be found in the monograph [66]) Kowalski and Ono studied some varieties of bounded integral commutative residuated lattices. In particular, they considered for every $n \geq 2$ the varieties defined by the following equations:

$$x^n \approx x^{n-1} \quad (E_n)$$

(E_2) corresponds, in fact, to the law of contraction, which defines the variety of Heyting algebras. Therefore, for every $n \geq 3$ the equation (E_n) corresponds to a weak form of contraction that we will call n -contraction.

In [27] Ciabattoni, Esteva and Godo brought the equations (E_n) to the framework of fuzzy logics. Indeed, for each $n \geq 2$, they defined the n -contraction axiom as:

$$\varphi^{n-1} \rightarrow \varphi^n \quad (C_n)$$

and they called C_n MTL (resp. C_n IMTL) the extension of MTL (resp. IMTL) obtained by adding this axiom.

Given $n \geq 2$, the equivalent algebraic semantics of C_n MTL (resp. C_n IMTL) is the class of n -contractive MTL-algebras (resp. IMTL-algebras), i.e. the subvariety of MTL (resp. IMTL) defined by the equation:

$$x^{n-1} \approx x^n$$

Strong standard completeness for these logics was also proved in [27]:

Theorem 8.1 ([27]). *For every $n \geq 3$, C_n MTL and C_n IMTL enjoy the SSC.*

It is easy to see that C_2 MTL is Gödel logic and C_2 IMTL is the classical propositional calculus. Moreover, for every $n \geq 3$, WNM is a strict extension of C_n MTL, NM is a strict extension of C_n IMTL, C_n MTL is a strict extension of C_{n+1} MTL and C_n IMTL is a strict extension of C_{n+1} IMTL (see Figure 8.1).

We say that an axiomatic extension of MTL L is n -contractive if, and only if, $\vdash_L (C_n)$. Of course, given any L we can make it n -contractive by adding the schema (C_n) . We call the resulting logic $C_n L$.

For n -contractive logics it is easy to improve the Local Deduction-Detachment Theorem to the following form of DDT:

Theorem 8.2. *If L is an n -contractive axiomatic extension of MTL, then for every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$ we have:
 $\Gamma, \varphi \vdash_L \psi$ if, and only if, $\Gamma \vdash_L \varphi^{n-1} \rightarrow \psi$.*

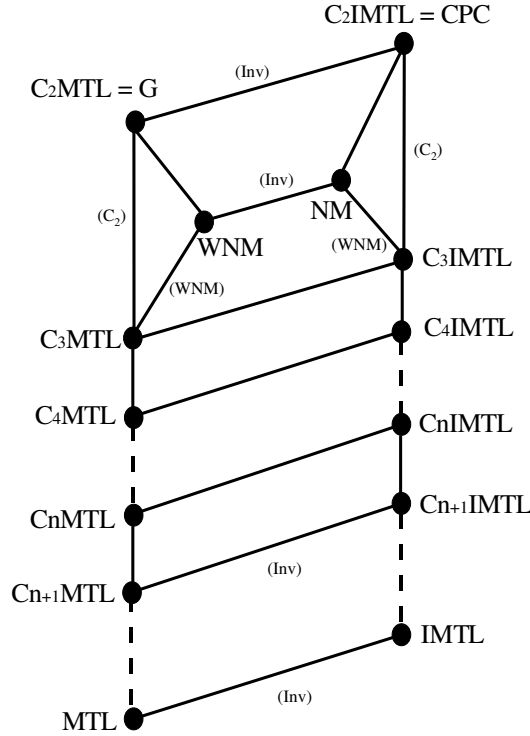


Figure 8.1: Graphic of axiomatic extensions of MTL obtained by adding all combinations of the schemata (Inv), (C_n) and (WNM). All the depicted inclusions are proper.

Moreover, in [104] Kowalski and Ono prove also the following result:

Proposition 8.3 (Prop 1.11, [104]). *Let \mathbb{K} be a variety of residuated lattices.*

Then, \mathbb{K} has the property *EDPC* (equationally definable principal congruences) if, and only if, $\mathbb{K} \models (E_n)$, for some $n \geq 2$.

According to Theorem 2.21, an algebraizable logic has the DDT if, and only if, its equivalent algebraic semantics has the EDPC. Therefore, in our framework of fuzzy logics as axiomatic extensions of MTL, the contractive logics are a good choice in the sense that they are the only finitary extensions of MTL enjoying the global Deduction-Detachment Theorem.

Finally, in [104] the following equations were also considered for every $n \geq 2$:

$$x \vee \neg x^{n-1} \approx \bar{1} \quad (EM_n)$$

Notice that (EM_2) is the algebraic form of the law of the excluded middle, and for every $n \geq 3$ (EM_n) corresponds to a weak form of this law.

We will consider also the axioms corresponding to (EM_n) :

$$\varphi \vee \neg \varphi^{n-1} \quad (S_n)$$

Given any axiomatic extension L of MTL, $S_n L$ will be its extension with (S_n) .

8.2 *n*-contractive chains

In this section we will study some basic properties of the *n*-contractive chains. First, observe that this is an important and big class of chains since it contains all the finite MTL-chains.

Proposition 8.4. *All finite MTL-chains are *n*-contractive for some *n*.*

Proof: Let \mathcal{A} be a finite MTL-chain with n elements. Take an arbitrary $a \in A \setminus \{\bar{1}^{\mathcal{A}}\}$. For every $i > 0$, $a^i \leq a^{i-1}$, thus necessarily $a^{n-1} = a^n$. \square

Proposition 8.5. *Let \mathcal{A} be an MTL-algebra and $a \in Id(\mathcal{A})$. Then for every $b, c \in A$,*

(1) *If $b, c \geq a$, then $b * c \geq a$.*

(2) *If $b \geq a$, then $a * b = a$.*

Proof: If $b, c \geq a$, then $a = a * a \leq b * c$. If $b \geq a$, then $a = a * a \leq a * b$, and the other inequality is always true. \square

The idempotent elements are easily described in *n*-contractive chains and, in addition, their number can be expressed equationally as the following propositions show.

Proposition 8.6. *Let \mathcal{A} be an *n*-contractive MTL-algebra. Then, $Id(\mathcal{A}) = \{a^{n-1} : a \in A\}$.*

Proof: If $a \in A$ is idempotent, then $a = a^2 = \dots = a^{n-1}$. Conversely, take any $a \in A$ and consider a^{n-1} . Then, $a^{n-1} * a^{n-1} = a^n * a^{n-2} = a^{n-1} * a^{n-2} = \dots = a^{n-1}$, so $a^{n-1} \in Id(\mathcal{A})$. \square

Definition 8.7. For every $n \geq 3$ and $k \geq 2$, we define the next formula:

$$I_k^n(x_0, \dots, x_k) := \bigvee_{i < k} (x_i^{n-1} \rightarrow x_{i+1}^{n-1}).$$

Proposition 8.8. For every $n \geq 3$, every $k \geq 2$ and every n -contractive MTL-chain \mathcal{A} the following are equivalent:

- (1) $\mathcal{A} \models I_k^n(x_0, \dots, x_k) \approx \bar{1}$.
- (2) $|Id(\mathcal{A})| \leq k$.

Proof: Suppose that $|Id(\mathcal{A})| > k$. Then we can take $a_0, \dots, a_k \in Id(\mathcal{A})$ such that $a_0 > a_1 > \dots > a_k$. Then for every i , $a_i = a_i^{n-1}$ and $a_{i+1} = a_{i+1}^{n-1}$, so $a_i^{n-1} \rightarrow a_{i+1}^{n-1} \neq \bar{1}^{\mathcal{A}}$ and the equation is not satisfied. Conversely, suppose $|Id(\mathcal{A})| \leq k$ and take arbitrary elements $a_0, \dots, a_k \in A$. $a_0^{n-1}, \dots, a_k^{n-1} \in Id(\mathcal{A})$, so there are $i < j \leq k$ such that $a_i^{n-1} = a_j^{n-1}$. Hence there is a $l < k$ such that $a_l^{n-1} \rightarrow a_{l+1}^{n-1} = \bar{1}^{\mathcal{A}}$. \square

In n -contractive chains we can also give a nice description of Archimedean classes:

Proposition 8.9. Let \mathcal{A} be an n -contractive MTL-chain. Then, for every $a, b \in A$:

- (i) $a \sim b$ if, and only if, $a^{n-1} = b^{n-1}$, and
- (ii) $a^{n-1} = \min[a]_{\sim}$.

Proof: (i) One direction is obvious. For the other one, suppose that $a \sim b$ and, for instance, $a \leq b$. Then there is $i \geq 1$ such that $b^i \leq a \leq b$, hence by the n -contraction law $b^{n-1} \leq a \leq b$. On one hand, we have $a^{n-1} \leq b^{n-1}$, since $a \leq b$. On the other hand, since b^{n-1} is an idempotent smaller than a using Proposition 8.5 we obtain $b^{n-1} \leq a^{n-1}$.

- (ii) It is clear that $a^{n-1} \in [a]_{\sim}$. Take an arbitrary $b \in [a]_{\sim}$. By (i), $a^{n-1} = b^{n-1}$, hence $b \geq b^{n-1} = a^{n-1}$. \square

Corollary 8.10. Let \mathcal{A} be an n -contractive MTL-chain and let $a \in A$. If $[a]_{\sim}$ has supremum, then it is the maximum.

Proof: Assume that b is the supremum of $[a]_{\sim}$. Then, $b^{n-1} = (\sup\{x \in A \mid x^{n-1} = a^{n-1}\})^{n-1} = \sup\{x^{n-1} \in A \mid x^{n-1} = a^{n-1}\} = a^{n-1}$, hence $b \in [a]_{\sim}$. \square

Therefore, Archimedean classes with supremum in n -contractive chains are always intervals of the form $[b^{n-1}, b]$. Moreover, this implies that given a standard n -contractive chain \mathcal{A} , in the set $Id(\mathcal{A})$ none of the elements has neither predecessor nor successor. In particular, 1 is an accumulation point of idempotent elements.¹

Next proposition characterizes the subdirectly irreducible n -contractive algebras.

¹These remarks on n -contractive left-continuous t-norms are already available in [114].

Proposition 8.11. *Let \mathcal{A} be an n -contractive MTL-chain. Then:*

\mathcal{A} is subdirectly irreducible if, and only if, the set of idempotent elements has a coatom.

Proof: First suppose that \mathcal{A} is subdirectly irreducible and let F be the minimum non-trivial filter. Given any $a \in F \setminus \{\bar{1}^{\mathcal{A}}\}$, it is clear that a^{n-1} is a coatom of $\text{Id}(\mathcal{A})$. Conversely, suppose a is the coatom in the set of idempotent elements. Then for every b such that $a < b < \bar{1}^{\mathcal{A}}$, we have $b^{n-1} = a$, so $[a, \bar{1}^{\mathcal{A}}]$ is the least non-trivial filter and \mathcal{A} is subdirectly irreducible. \square

Corollary 8.12. *There are no subdirectly irreducible standard n -contractive MTL-chains.*

Nevertheless, notice that this does not contradict the fact that the varieties $\mathbb{C}_n\text{MTL}$ and $\mathbb{C}_n\text{IMTL}$ are generated by their standard chains.

The generalized excluded middle equations (EM_n) describe exactly the simple n -contractive chains.

Proposition 8.13. *Let \mathcal{A} be an MTL-chain. The following are equivalent:*

- (i) $\mathcal{A} \models (EM_n)$.
- (ii) \mathcal{A} is n -contractive and simple.

Proof: (i) \Rightarrow (ii) : If $\mathcal{A} \models (EM_n)$, then for every $a \in A \setminus \{\bar{1}^{\mathcal{A}}\}$, $a^{n-1} = \bar{0}^{\mathcal{A}}$. Therefore, for every $a \in A \setminus \{\bar{1}^{\mathcal{A}}\}$, $a^{n-1} = a^n$, and it is clearly simple. (ii) \Rightarrow (i) : If \mathcal{A} is n -contractive and simple, then for $a \in A$, a^{n-1} is idempotent. Therefore, if $a \neq \bar{1}^{\mathcal{A}}$, then $a^{n-1} = \bar{0}^{\mathcal{A}}$, and hence $\mathcal{A} \models (EM_n)$. \square

Recall that an algebra is semisimple if, and only if, it is representable as a subdirect product of simple algebras. For an MTL-algebra \mathcal{A} this is equivalent to $\text{Rad}(\mathcal{A}) = \{\bar{1}^{\mathcal{A}}\}$.

Corollary 8.14. *For each $n \geq 2$, the class of semisimple n -contractive MTL-algebras is the variety $\mathbb{S}_n\text{MTL}$.*

For MV-algebras those varieties are easy to describe:

Lemma 8.15. *Let \mathcal{A} be an MV-chain. The following are equivalent:*

- (i) $\mathcal{A} \models (E_n)$.
- (ii) $\mathcal{A} \in \mathbf{I}(\{L_1, \dots, L_n\})$.
- (iii) $\mathcal{A} \models (EM_n)$.

Corollary 8.16. *For each $n \geq 2$, $\mathbb{S}_n\text{MTL} \cap \text{MV} = \mathbb{C}_n\text{MTL} \cap \text{MV} = \mathbf{V}(\{L_1, \dots, L_n\})$.*

However, in MTL and in IMTL the situation is not so easy. In the first level the varieties corresponding to (E_n) and (EM_n) are still easy to compute. Indeed, $S_2\text{MTL} = S_2\text{IMTL} = C_2\text{IMTL} = \mathbf{BA}$ and $C_2\text{MTL} = \mathbf{G}$. For $n = 3$, we have $S_3\text{MTL} \subsetneq \mathbf{WNM}$, in fact, the $S_3\text{MTL}$ -chains are those where the product of two non-one elements is always zero, i. e. the so-called drastic product. When $n = 3$, we also have $S_3\text{IMTL} = \mathbf{V}(\mathbf{L}_3) = \mathbf{NM} \cap \mathbf{MV} \subsetneq \mathbf{NM} \subseteq C_3\text{IMTL}$. Therefore for each $n \geq 3$, $S_n\text{IMTL} \subsetneq C_n\text{IMTL}$ and $S_n\text{MTL} \subsetneq C_n\text{MTL}$. The variety $S_4\text{IMTL}$ and its lattice of subvarieties have been studied in [72].

Now we will show that the varieties of semisimple n -contractive chains are discriminator varieties.

Definition 8.17. For every $n \geq 3$ we define $\delta_n(x, y) := (x \leftrightarrow y)^{n-1}$.

Notice that if a, b are elements in a simple n -contractive MTL-chain, then:

- $a = b$ if, and only if, $\delta_n(a, b) = \bar{1}^A$.
- $a \neq b$ if, and only if, $\delta_n(a, b) = \bar{0}^A$.

With this term we can define a discriminator just by considering $t(x, y, z) := (\delta(x, y) \wedge z) \vee (\neg \delta(x, y) \wedge x)$. It is clear that $t(x, y, z) = x$ if $x \neq y$, and $t(x, y, z) = z$ otherwise. In fact, Kowalski has proved that the only discriminator varieties of bounded integral commutative residuated lattices are those satisfying some of the (EM_n) equations:

Theorem 8.18 ([105]). For every variety \mathbb{K} of bounded integral commutative residuated lattices, the following are equivalent:

- (i) $\mathbb{K} \models (EM_n)$ for some $n \geq 2$;
- (ii) \mathbb{K} is semisimple;
- (iii) \mathbb{K} is a discriminator variety.

Using δ one can also give an equational definition of the class of algebras without negation fixpoint:

Proposition 8.19. Let \mathcal{A} be a simple n -contractive MTL-chain. Then, \mathcal{A} has not the negation fixpoint if, and only if, $\mathcal{A} \models \delta_n(x, \neg x) \approx \bar{0}$.

Finally, we will study the existence of atoms and coatoms in n -contractive chains.

Proposition 8.20. Let \mathcal{A} be a simple n -contractive IMTL-chain and take $a_1, \dots, a_{n-2} \in A \setminus \{\bar{1}^A\}$. If $a_1 \& \dots \& a_{n-2} \neq \bar{0}^A$, then $a_1 \& \dots \& a_{n-2} = \min A \setminus \{\bar{0}^A\}$.

Proof: Given any $c \neq \bar{0}^A$ we must prove $a_1 \& \dots \& a_{n-2} \leq c$. Take $d := a_1 \vee \dots \vee a_{n-2} \vee \neg c \neq \bar{1}^A$, so $d^{n-1} = \bar{0}^A$. Then, $a_1 \& \dots \& a_{n-2} \rightarrow c = \neg(a_1 \& \dots \& a_{n-2} \& \neg c) \geq \neg d^{n-1} = \bar{1}^A$. \square

Corollary 8.21. *For every simple n -contractive non-trivial IMTL-chain \mathcal{A} , there is $a \in A$ such that $a = \max A \setminus \{\bar{1}^{\mathcal{A}}\}$ and $\neg a = \min A \setminus \{\bar{0}^{\mathcal{A}}\}$.*

Proposition 8.22. *Every simple n -contractive non-trivial MTL-chain \mathcal{A} has a coatom.*

Proof: Suppose there is no coatom. Hence, $\bigvee_{a < \bar{1}^{\mathcal{A}}} a = \bar{1}^{\mathcal{A}}$. But then $\bar{0}^{\mathcal{A}} = \bigvee_{a < \bar{1}^{\mathcal{A}}} a^{n-1} = (\bigvee_{a < \bar{1}^{\mathcal{A}}} a)^{n-1} = \bar{1}^{\mathcal{A}}$. \square

This implies that the n -contractive MTL-chains defined by a left-continuous t -norm are not simple, i. e. there are no standard S_n MTL-chains, which we already knew from the fact that there are not even subdirectly irreducible n -contractive standard chains.

8.3 Combining weakly cancellative and n -contractive fuzzy logics

In Chapter 7 the variety WCMTL of weakly cancellative MTL-algebras was defined to provide examples of indecomposable MTL-chains. Besides, the Ω operator gave rise to the variety $\Omega(\text{WCMTL})$ which was a kind of analog of \mathbb{BL} in the sense that here all the chains were also decomposable as ordinal sums of weakly cancellative semihoops. Now it seems natural to consider the intersection of these varieties with the classes of n -contractive algebras (or equivalently the supremum of the corresponding logics) in order to obtain some new kinds of algebras with a nice and simpler structure. Therefore, we will consider for every $n \geq 2$ the logics $S_n\text{WCMTL}$ and $C_n\text{WCMTL}$.

Proposition 8.23. *For every $n \geq 2$, $\{(WC), (C_n)\} \vdash_{\text{MTL}} (S_n)$.*

Proof: Let \mathcal{A} be an MTL-chain satisfying (WC) and (C_n) . We will prove that it is simple. Take an arbitrary $a \in A \setminus \{\bar{1}^{\mathcal{A}}\}$. Then, a^{n-1} is an idempotent element, hence $a^{n-1} \& a^{n-1} = a^{n-1} \& \bar{1}^{\mathcal{A}} = a^{n-1}$. Since the chain is weakly cancellative, this implies $a^{n-1} = \bar{0}^{\mathcal{A}}$. Therefore, \mathcal{A} is simple, i. e. satisfies (S_n) . \square

Corollary 8.24. *Given any axiomatic extension L of WCMTL and $n \geq 2$, the extensions obtained by (S_n) and (C_n) coincide. In particular, $S_n\text{WCMTL} = C_n\text{WCMTL}$.*

It is straightforward to prove that the Ω operator and the schemata (C_n) commute:

Proposition 8.25. *Let L be an axiomatic extension of MTL. For every $n \geq 2$, $\Omega(C_n L) = C_n \Omega(L)$.*

Therefore, we have $C_n \Omega(\text{WCMTL}) = \Omega(C_n \text{WCMTL}) = \Omega(S_n \text{WCMTL})$.

Finally, we will consider for every $n \geq 2$ the logic $\Omega(S_n \text{MTL})$ and we will show that it is also finitely axiomatizable.

Proposition 8.26. *Let \mathcal{A} be an n -contractive MTL-chain. The following are equivalent:*

- (i) \mathcal{A} is totally decomposable.
- (ii) \mathcal{A} is an ordinal sum of simple n -contractive chains.
- (iii) $\mathcal{A} \models (y^{n-1} \rightarrow x) \vee (x \rightarrow x \& y) \approx \bar{1}$.

Proof: (i) \Rightarrow (ii): If \mathcal{A} is decomposable as the ordinal sum of its Archimedean classes, then, by Proposition 8.9, it is decomposable as ordinal sum of simple chains.

(ii) \Rightarrow (iii): Take arbitrary elements $a, b \in A$. If $b \leq a$, then $b^{n-1} \leq a$, so they satisfy the equation. Suppose $a < b$. If they are in different components of the ordinal sum, then $a \rightarrow a \& b = \bar{1}^{\mathcal{A}}$. If they are in the same component, then, by simplicity, $b^{n-1} \leq a$.

(iii) \Rightarrow (i): Suppose that \mathcal{A} satisfies the equation. Take $a, b \in A \setminus \{\bar{1}^{\mathcal{A}}\}$ such that $a < b$ and they belong to different Archimedean classes. Then $b^{n-1} \rightarrow a \neq \bar{1}^{\mathcal{A}}$, so $a \rightarrow a \& b = \bar{1}^{\mathcal{A}}$, i. e. $a \& b = a$. Therefore, \mathcal{A} is the ordinal sum of its Archimedean classes. \square

Corollary 8.27. $\Omega(\mathbb{S}_n\text{MTL})$ is the variety generated by the totally decomposable n -contractive chains, and it is axiomatized by $(y^{n-1} \rightarrow x) \vee (x \rightarrow x \& y) \approx \bar{1}$.

8.4 Some properties of n -contractive fuzzy logics

In this section we will study some logical and algebraic properties of the considered logics. First we focus our attention on local finiteness. n -contractivity is a necessary condition for local finiteness:

Proposition 8.28. *Let $\mathbb{K} \subseteq \text{MTL}$ be a variety. If \mathbb{K} is locally finite, then there exists some $n \geq 2$ such that $\mathbb{K} \models x^n \approx x^{n-1}$.*

Proof: Suppose that for every $n \geq 2$, there is $\mathcal{A}_n \in \mathbb{K}$ and $a_n \in A_n$ such that $a_n^n < a_n^{n-1}$. Consider the algebra $\prod_{n \geq 2} \mathcal{A}_n$ and the element $a = \langle a_2, a_3, a_4, \dots \rangle \in \prod_{n \geq 2} A_n$. Then for every $n \geq 2$, we have $a^n < a^{n-1}$, thus the subalgebra generated by a is infinite. \square

However, the sufficiency of this condition remains as an open problem, so for each variety of n -contractive MTL-algebras we have to discuss whether it is locally finite or not. It is straightforward that to show that \mathbb{G} is locally finite. The property has been proved true also for NMI in [71], and we have generalized the result to WNMI in Chapter 9.

Theorem 8.29. $\Omega(\mathbb{S}_3\text{MTL})$ is locally finite. (This case is isomorphic to the case of WNMI).

Finally, $\mathbb{S}_4\text{IMTL}$ is also proved to be locally finite in [72]. The property is still unknown for any variety of contractive MTL-algebras not contained in the previous ones.

We turn now to the finite embedding property. Ono² and Montagna et al in [28] proved it not only for MTL, IMTL and SMTL, but also for $\mathbb{C}_n\text{MTL}$ and $\mathbb{C}_n\text{IMTL}$, for every $n \geq 2$. This allows us to prove the following result:

Proposition 8.30. $\text{MTL} = \bigvee_{n \geq 2} \mathbb{C}_n\text{MTL}$ and $\text{IMTL} = \bigvee_{n \geq 2} \mathbb{C}_n\text{IMTL}$.

Proof: MTL and IMTL have the FEP, therefore they are generated by their finite chains. Since all finite MTL-chains are n -contractive for some n , we obtain that MTL is generated by the contractive MTL-chains and IMTL is generated by the contractive IMTL-chains. \square

Theorem 8.31. *For every $n \geq 2$, the following varieties have the FEP:*

- $\mathbb{S}_n\text{MTL}$
- $\Omega(\mathbb{S}_n\text{MTL})$
- $\mathbb{C}_n\text{WCMTL}$
- $\Omega(\mathbb{C}_n\text{WCMTL})$

Proof: Let \mathbb{L} be any variety from the class $\{\mathbb{S}_n\text{MTL}, \Omega(\mathbb{S}_n\text{MTL}), \mathbb{C}_n\text{WCMTL}, \Omega(\mathbb{C}_n\text{WCMTL}) \mid n \geq 2\}$. Let \mathcal{A} be an arbitrary L-chain and $B \subseteq A$ be a finite subset of its carrier. Consider the monoid \mathcal{M} generated by $B \cup \{\bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}}\}$, i. e. the submonoid of $\langle A, \&, \bar{1}^{\mathcal{A}} \rangle$ obtained by closing $B \cup \{\bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}}\}$ under $\&$. By the simplicity of \mathcal{A} , \mathcal{M} is finite, so it is residuated. Expanding \mathcal{M} with the residuum it becomes an MTL-chain. Observe that, in fact, \mathcal{M} is an L-chain, since the required properties are preserved when we generate the monoid. Finally, it is clear that B can be embedded in \mathcal{M} , thus \mathbb{L} has the FEP. \square

Open problems: FEP, FMP and decidability for $\mathbb{S}_n\text{IMTL}$.

Finally, we consider the standard completeness of n -contractive fuzzy logics. As mentioned above, the SSC was proved by Ciabattoni, Esteva and Godo in [27] for $\mathbb{C}_n\text{MTL}$ and $\mathbb{C}_n\text{IMTL}$ for every $n \geq 2$.

Theorem 8.32. *For every $n \geq 2$, we have:*

- (a) $\mathbb{S}_n\text{MTL}$, $\mathbb{S}_n\text{IMTL}$ and $\mathbb{S}_n\text{WCMTL}$ do not enjoy SC because there are no standard algebras in the corresponding variety.
- (b) $\Omega(\mathbb{S}_n\text{MTL})$ enjoys the SSC.

²Private communication.

Proof: (a): It follows from Proposition 8.22.

(b): We will prove it by using the embedding method of Jenei and Montagna (see [100]). Let \mathcal{A} be a countable $\Omega(S_n\text{MTL})$ -chain. Consider the following set:

$X := \{\langle s, q \rangle : s \in A, s \neq \bar{0}^A, q \in Q \cap (0, 1]\} \cup \{\langle \bar{0}^A, 1 \rangle\}$, equipped with the lexicographical order and the operation

$$\langle s, q \rangle \circ \langle s', q' \rangle := \begin{cases} \min\{\langle s, q \rangle, \langle s', q' \rangle\} & \text{if } s \& s' = \min\{s, s'\} \\ \langle s \& s', 1 \rangle & \text{otherwise.} \end{cases}$$

Let $\langle s, q \rangle, \langle s', q' \rangle \in X$ be such that $\langle s, q \rangle < \langle s', q' \rangle^{n-1}$. We must prove that $\langle s, q \rangle \circ \langle s', q' \rangle = \langle s, q \rangle$.

We have:

$$\langle s', q' \rangle^{n-1} = \begin{cases} \min\{\langle s', q' \rangle\} & \text{if } (s')^2 = s' \\ \langle (s')^{n-1}, 1 \rangle & \text{otherwise.} \end{cases}$$

Therefore, $s \leq s'$. If $s \& s' = s$, we are done. Suppose that $s \& s' < s$. If $s' \& s' = s'$, then, since \mathcal{A} is totally decomposable, we have $s \& s' = s$; a contradiction. If $s' \& s' \neq s'$, then $\langle s', q' \rangle^{n-1} = \langle (s')^{n-1}, 1 \rangle$, thus $s \leq (s')^{n-1}$ and this implies that $s \& s' = s$; a contradiction.

Consider now the completion of the operation in $[0, 1]$: for every $a, b \in [0, 1]$, define $a \otimes b := \sup\{q \circ p : q \leq a, p \leq b, q, p \in Q\}$. Let $a, b \in [0, 1]$ be such that $a \leq b^{n-1}$. We must prove that $a \otimes b = a$. It is clear that $\sup\{q \circ p : q \leq a, p \leq b, q < p^{n-1}, q, p \in Q\} = a$, so it is enough to proof:

$$\sup\{q \circ p : q \leq a, p \leq b, q, p \in Q\} = \sup\{q \circ p : q \leq a, p \leq b, q < p^{n-1}, q, p \in Q\}.$$

It is obvious that the second member of the equality is smaller or equal than the first one. Let us prove the other inequality. Suppose $q \leq a, p \leq b, q, p \in Q$. If $q < p^{n-1}$, we are done. Suppose not. Then take $b \geq p' > p$ such that $q < (p')^{n-1}$. \square

Open problem: Standard completeness for $\Omega(C_n\text{WCMTL})$.

8.5 Conclusions

A new hierarchy of fuzzy logics has been defined in this chapter by using the axioms of n -contraction, the generalized excluded middle axioms, the weak cancellation axioms and the Ω operator. We have studied some of their logical and algebraic properties. The obtained results are gathered in Table 8.1 where the remaining open problems are also highlighted.

Table 8.1: Logical and algebraic properties of n -contractive fuzzy logics

Logic	LF	FEP	FMP	Decidable	SC	FSSC	SSC
G	Yes	Yes	Yes	Yes	Yes	Yes	Yes
WNM	Yes	Yes	Yes	Yes	Yes	Yes	Yes
NM	Yes	Yes	Yes	Yes	Yes	Yes	Yes
C_n MTL	?	Yes	Yes	Yes	Yes	Yes	Yes
C_n IMTL	?	Yes	Yes	Yes	Yes	Yes	Yes
S_n MTL	?	Yes	Yes	Yes	No	No	No
S_n IMTL	?	?	?	?	No	No	No
$\Omega(S_n$ MTL)	?	Yes	Yes	Yes	Yes	Yes	Yes
S_n WCMTL	?	Yes	Yes	Yes	No	No	No
$\Omega(C_n$ WCMTL)	?	Yes	Yes	Yes	?	?	?

Chapter 9

The variety of Weak Nilpotent Minimum algebras

After the general results on varieties of n -contractive MTL-algebras given in the previous chapter, now we focus on a particular kind of 3-contractive algebras, namely the Weak Nilpotent Minimum algebras. The reason is that their structure is quite simple and thus we are able to present several results on the axiomatization of their subvarieties, local finiteness and standard completeness properties. First, we survey the known results for their involutive members, NM-algebras, and later we move to the more general case of WNM-algebras.

9.1 Varieties of NM-algebras

The lattice of subvarieties of NM has been completely described in [71]. We will briefly present this description.

The structure of finite NM-chains is very simple. In fact, for every $n \geq 1$ there is exactly one, up to isomorphism, NM-chain with n elements. Therefore, we can consider the following canonical finite NM-chains.

For every $n \geq 1$ the canonical NM-chain of $2n$ elements is defined as $\mathcal{N}_{2n} := \langle \{-n, -(n-1), \dots, -1, 1, \dots, n-1, n\}, \&, \rightarrow, \wedge, \vee, -n, n \rangle$ and the canonical NM-chain of $2n+1$ elements is defined as $\mathcal{N}_{2n+1} := \langle \{-n, -(n-1), \dots, -1, 0, 1, \dots, n-1, n\}, \&, \rightarrow, \wedge, \vee, -n, n \rangle$, where:

$$a \& b := \begin{cases} \min\{a, b\} & \text{if } a > -b, \\ -n & \text{otherwise.} \end{cases}$$
$$a \rightarrow b := \begin{cases} n & \text{if } a \leq b, \\ \max\{-a, b\} & \text{otherwise.} \end{cases}$$

$a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

Recall also the definition of the unique (up to isomorphism) standard NM-chain, $[0, 1]_{\text{NM}}$, given in Chapter 3.

Notice that all these chains, in fact all NM-chains, are perfect or perfect plus the fixpoint. Therefore, $\text{NM} \subseteq \text{IBP}_0^{+1}$.

Given an NM-chain \mathcal{C} with fixpoint, we denote by \mathcal{C}^- the subalgebra obtained by erasing the fixpoint. With this notation, it is clear that $\mathcal{N}_{2n} = \mathcal{N}_{2n+1}^-$ for every $n \geq 1$.

Theorem 9.1 ([71]). *A variety of NM-algebras is a proper subvariety of NM if, and only if, it does not contain \mathcal{N}_n for some $n \geq 1$.*

Corollary 9.2 ([71]). *If \mathcal{A} is an infinite NM-chain with fixpoint, then $\mathbf{V}(\mathcal{A}) = \text{NM}$.*

Theorem 9.3 ([71]). *NM is locally finite.*

This fact, as already discussed, implies the FMP and the decidability of NM, thus in particular we have that every variety of NM-chains is generated by its finite chains. It leads to the following classification of the subvarieties of NM:

Theorem 9.4 ([71]). *Every proper subvariety of NM is:*

1. $\mathbf{V}([0, 1]_{\text{NM}}^-) = \mathbf{V}(\{\mathcal{N}_{2n} : n \geq 1\})$ or
2. $\mathbf{V}(\mathcal{N}_{2n+1})$ or
3. $\mathbf{V}(\mathcal{N}_{2n})$ or
4. $\mathbf{V}([0, 1]_{\text{NM}}^-, \mathcal{N}_{2n+1})$ or
5. $\mathbf{V}(\mathcal{N}_{2n}, \mathcal{N}_{2m+1})$, with $m < n$.

Furthermore, equational bases for these varieties are obtained by means of the following terms: $S_n(x_0, \dots, x_n) := \bigwedge_{i < n} ((x_i \rightarrow x_{i+1}) \rightarrow x_{i+1}) \rightarrow \bigvee_{i < n+1} x_i$, for every $n \geq 2$.

Theorem 9.5 ([71]). *Let \mathcal{A} be an NM-chain. Then:*

1. $\mathcal{A} \models S_n(x_0, \dots, x_n) \approx \bar{1}$ if, and only if, it has less than $2n + 2$ elements.
2. $\mathcal{A} \models Bp(x) \approx \bar{1}$ if, and only if, it has no fixpoint.

Corollary 9.6 ([71]). *The proper subvarieties of NM admit the following axiomatizations (relative to NM):*

1. $\mathbf{V}([0, 1]_{\text{NM}}^-)$ is axiomatized by $Bp(x) \approx \bar{1}$.
2. $\mathbf{V}(\mathcal{N}_{2n+1})$ is axiomatized by $S_n(x_0, \dots, x_n) \approx \bar{1}$.
3. $\mathbf{V}(\mathcal{N}_{2n})$ is axiomatized by $S_n(x_0, \dots, x_n) \approx \bar{1}$ and $Bp(x) \approx \bar{1}$.

9.2 Weak nilpotent minimum algebras

Our aim now is to generalize the above results to the whole variety of WNM-algebras.

The operations in WNM-chains are very simple as the following lemma states:

Lemma 9.7. *Let $\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \bar{0}, \bar{1} \rangle$ be a WNM-chain. Then for every $a, b \in A$:*

$$a \& b = \begin{cases} a \wedge b & \text{if } a > \neg b, \\ \bar{0}^{\mathcal{A}} & \text{otherwise.} \end{cases}$$

$$a \rightarrow b = \begin{cases} \bar{1}^{\mathcal{A}} & \text{if } a \leq b, \\ \neg a \vee b & \text{otherwise.} \end{cases}$$

Notice that the previous lemma generalizes the structure of standard WNM-chains presented in Chapter 4. It turns out, that WNM-chains depend essentially on the negation operation, thus we need to recall some properties of such operations.

Lemma 9.8. *Let \mathcal{A} be a WNM-chain. Then for every $a \in A$:*

- (i) $\neg a = \neg \neg \neg a$,
- (ii) $a \leq \neg \neg a$,
- (iii) $a = \neg \neg a$ if, and only if, there is $b \in A$ such that $a = \neg b$, and
- (iv) $\neg \neg a = \min\{b \in A : a \leq b \text{ and } b = \neg \neg b\}$.

The last one gives rise to the following definition:

Definition 9.9. *Let \mathcal{A} be a WNM-chain and let $a \in A$ be an involutive element. We define $I_a^{\mathcal{A}} := \{b \in A : \neg b = \neg a\}$ and we call it the interval associated to a , where the negation function is constant with value $\neg a$. We say that a has a trivial associated interval when $I_a^{\mathcal{A}} = \{a\}$. When \mathcal{A} is a standard WNM-chain given by a t -norm $*$, we will sometimes write I_a^* instead of $I_a^{\mathcal{A}}$. We will write just I_a when the algebra is clear from the context.*

Now we can define the finite partition property for WNM-chains.

Definition 9.10. *Let \mathcal{A} be a WNM-chain and consider its negation operation $\neg^{\mathcal{A}}$. We say that \mathcal{A} has a finite partition iff $\neg^{\mathcal{A}}$ is constant in a finite number of intervals, i.e. the set $\{a \in A : I_a \neq \{a\}\}$ is finite. Let I_{a_1}, \dots, I_{a_n} be these intervals. In such a case we define the associated finite partition P defined in the following way:*

- $I_{a_1}, \dots, I_{a_n} \in P$.

- Consider the set $X = A \setminus (I_{a_1} \cup \dots \cup I_{a_n})$. It is clear that all the elements in X are involutive. For every connected component Y of $X \cap A_-$, consider the elements $\neg a_{i_1} < \dots < \neg a_{i_k} \in Y$, and then add every interval $Y \cap [0, \neg a_{i_1}]$, $(\neg a_{i_1}, \neg a_{i_2}]$, \dots , $(\neg a_{i_{k-1}}, \neg a_{i_k}]$, $Y \cap [\neg a_{i_k}, 1]$ to P . If there are no elements of the form $\neg a_i$ in Y , we add Y to P . We do the same for every connected component Y of $X \cap A_+$.

Notice that this partition yields two kinds of intervals: those where the negation takes a constant value, and those where all elements are involutive. As a matter of nomenclature, we call them *constant intervals* and *involutive intervals*, respectively. Figure 9.2 shows an example of a WNM t-norm with a fixpoint, a_3 , and with a finite partition where the constant intervals are $[a_4, a_5]$ and $[a_6, a_7]$, while the involutive intervals are $[0, a_1]$, $(a_1, a_2]$, $(a_2, a_3]$, (a_3, a_4) , (a_5, a_6) and $(a_7, 1]$.

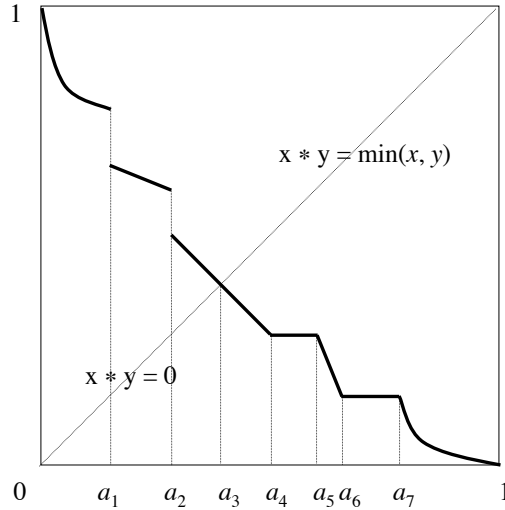


Figure 9.2: An example of WNM t-norm with a finite partition.

Figure 9.3 shows three families of WNM t-norms with finite partition parametrized with a real number c : $c \in [0, 1)$ for \otimes_c , $c \in [1/2, 1)$ for \star_c and $c \in [1/2, 1]$ for \odot_c . Notice that $\otimes_0 = \odot_1 = \min$ and $\star_{1/2} = \odot_{1/2}$ is the Nilpotent Minimum t-norm. These families are actually the only WNM t-norms with a finite partition of at most three intervals.

To refer to the class of WNM t-norms and those with a finite partition we will use from now on the following notation:

$$\mathbf{WNM} = \{ * \text{ is a weak nilpotent minimum t-norm} \}$$

$$\mathbf{WNM}\text{-}\mathbf{fin} = \{ * \in \mathbf{WNM} \mid * \text{ has a finite partition} \}$$

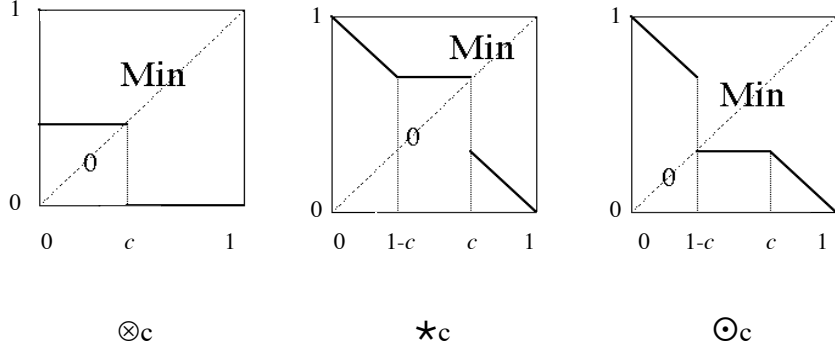


Figure 9.3: Three parametric families of WNM t-norms with finite partition.

Definition 9.11. Let \mathcal{A} be a WNM-chain. $N(\mathcal{A})$ will denote the set of involutive elements of \mathcal{A} , i.e. $N(\mathcal{A}) = \{\neg a : a \in \mathcal{A}\}$.

Proposition 9.12. Let \mathcal{A} be a WNM-chain. Then $N(\mathcal{A})$ is the universe of the maximum NM-subalgebra of \mathcal{A} . We denote it by $\mathcal{N}(\mathcal{A})$.

Proof: We must prove that $N(\mathcal{A})$ is closed under all operations. Obviously, $0 = \neg 1 \in N(\mathcal{A})$ and $1 = \neg 0 \in N(\mathcal{A})$. Take $\neg a, \neg b \in N(\mathcal{A})$. Since \mathcal{A} is linearly ordered, $\neg a \wedge \neg b, \neg a \vee \neg b \in N(\mathcal{A})$, hence $\neg a \& \neg b \in N(\mathcal{A})$. Finally, if $\neg a \leq \neg b$, then $\neg a \rightarrow \neg b = 1 \in N(\mathcal{A})$; otherwise $\neg a \rightarrow \neg b = \neg \neg a \vee \neg b \in N(\mathcal{A})$. \square

Proposition 9.13. Let $\mathbb{K} \subseteq \mathbf{WNM}$ be a variety. Then, $\mathbb{K} \cap \mathbf{NM} = \mathbf{V}(\{\mathcal{N}(\mathcal{A}) : \mathcal{A} \text{ chain of } \mathbb{K}\})$.

Proof: The inclusion from right to left is clear, since for every chain of \mathbb{K} , \mathcal{A} , we have that $\mathcal{N}(\mathcal{A})$ is an NM-chain and it is a subalgebra of an algebra in \mathbb{K} , so $\mathcal{N}(\mathcal{A}) \in \mathbb{K} \cap \mathbf{NM}$. Conversely, if \mathcal{C} is a chain of $\mathbb{K} \cap \mathbf{NM}$, then $\mathcal{C} = \mathcal{N}(\mathcal{C}) \in \{\mathcal{N}(\mathcal{A}) : \mathcal{A} \text{ chain of } \mathbb{K}\}$, and by the subdirect representation theorem, we obtain the inclusion. \square

Now we can prove that the variety of WNM-algebras is locally finite.

Lemma 9.14. Let \mathcal{A} be a WNM-chain. Then, every finite subset of \mathcal{A} generates a finite WNM-chain.

Proof: Take a finite subset $B = \{b_0, \dots, b_n\} \subseteq \mathcal{A}$. Due to Lemma 9.7 and (i) of Lemma 9.8 it is obvious that the universe of the subalgebra generated by B is $\{0^{\mathcal{A}}, 1^{\mathcal{A}}, b_0, \dots, b_n, \neg b_0, \dots, \neg b_n, \neg \neg b_0, \dots, \neg \neg b_n\}$, so it is finite. \square

Proposition 9.15. *WNM is a locally finite variety.*

Proof: Let \mathcal{A} be a WNM-algebra and take a finite set $B \subseteq A$. Suppose that $B = \{b_0, \dots, b_n\}$. We must prove that $\langle B \rangle_{\mathcal{A}}$ is also finite. If \mathcal{A} is a chain, we apply the previous lemma. Suppose that \mathcal{A} is not a chain. Then, due to the theorem of representation in subdirect products of chains, we have an embedding $\alpha : \mathcal{A} \hookrightarrow \prod_{i \in I} \mathcal{A}_i$, where for every $i \in I$, \mathcal{A}_i is a WNM-chain. Consider the images of the elements of B , $\alpha(b_j) = (a_i^j)_{i \in I}$, for every $j \in \{1, \dots, n\}$. We have seen that for every $i \in I$, $\{a_i^1, \dots, a_i^n\}$ generates a finite chain $\mathcal{C}_i \subseteq \mathcal{A}_i$ whose universe is $\{0^{\mathcal{A}_i}, 1^{\mathcal{A}_i}, a_i^1, \dots, a_i^n, \neg a_i^1, \dots, \neg a_i^n, \neg\neg a_i^1, \dots, \neg\neg a_i^n\}$. Notice that there is only a finite number of such chains up to isomorphism, say $\{\mathcal{C}_0, \dots, \mathcal{C}_{n-1}\}$, and $\langle B \rangle_{\mathcal{A}} \in \mathbf{V}(\{\mathcal{C}_i : i < n\})$. Therefore, using that every variety generated by a finite number of finite algebras is locally finite ([24], Theorem 10.16), we obtain that $\langle B \rangle_{\mathcal{A}}$ is finite. \square

We have the following easy consequences:

- WNM has the FEP.
- WNM has the FMP.
- $\text{WNM} = \mathbf{V}(\text{WNM}_{fin}) = \mathbf{Q}(\text{WNM}_{fin})$.
- Every subvariety of WNM is generated by its finite chains.
- WNM and all its axiomatic extensions are decidable.

Lemma 9.16. *Let \mathcal{A} be a WNM-chain, let $F \in \text{Fi}(\mathcal{A})$ and consider the quotient algebra \mathcal{A}/F . Then:*

- $[\bar{1}^{\mathcal{A}}]_F = F$
- $[\bar{0}^{\mathcal{A}}]_F = \bar{F}$
- For every $a, b \in A \setminus (F \cup \bar{F})$ such that $a \neq b$, we have $[a]_F \neq [b]_F$.

Proof: The first statement is trivial. As for the second, take an arbitrary $a \in A$. Then, $a \in [\bar{0}^{\mathcal{A}}]_F$ iff $a \rightarrow \bar{0}^{\mathcal{A}} \in F$ iff $a \in \bar{F}$. Now consider a pair of different elements $a, b \in A \setminus (F \cup \bar{F})$. Suppose, for instance, that $a > b$. Then, $a \rightarrow b = \neg a \vee b \notin F$, hence $[a]_F \neq [b]_F$. \square

Lemma 9.17. *Let \mathcal{A} and \mathcal{B} be WNM-chains and let $f : \mathcal{A} \rightarrow \mathcal{B}$ a surjective homomorphism. Then:*

- (i) *If $I_{\bar{1}^{\mathcal{B}}} = \{\bar{1}^{\mathcal{B}}\}$, then \mathcal{B} is embeddable in \mathcal{A} .*
- (ii) *If $I_{\bar{1}^{\mathcal{B}}} \neq \{\bar{1}^{\mathcal{B}}\}$, then there is $a \in N(\mathcal{A}) \cap A_+$ such that $I_a \neq \{a\}$ and \mathcal{B} is embeddable in \mathcal{A}/F^a .*

Proof: By the Homomorphism Theorem we know that $\mathcal{A}/\text{Ker } f \cong \mathcal{B}$, thus, after the previous lemma, we can assume that the carrier of \mathcal{B} is $(\mathcal{A} \setminus (F \cup \overline{F})) \cup \{\overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\}$. (i) is obvious. Assume that $I_{\overline{1}^{\mathcal{B}}} \neq \{\overline{1}^{\mathcal{B}}\}$. Take $c \in I_{\overline{1}^{\mathcal{B}}} \setminus \{\overline{1}^{\mathcal{B}}\}$, then it is clear that \mathcal{B} is embeddable in $\mathcal{A}/F^{\neg \neg c}$. \square

Corollary 9.18. *Let \mathcal{A} be a WNM-chain. Then, $\mathbf{H}(\mathcal{A}) = \mathbf{IS}(\mathcal{A}) \cup \mathbf{IS}(\{\mathcal{A}/F^a : a \in N(\mathcal{A}) \cap A_+ \text{ and } I_a \neq \{a\}\})$. Moreover, if there exists the maximum positive involutive element a such that for any other $b \in N(\mathcal{A}) \cap A_+$, $|I_b| \leq |I_a|$, then $\mathbf{H}(\mathcal{A}) = \mathbf{IS}(\mathcal{A}) \cup \mathbf{IS}(\mathcal{A}/F^a)$.*

Notice that for every standard WNM-chain $[0, 1]_*$ whose t-norm is in **WNM-fin**, there is a maximum positive involutive element a such that $I_a \neq \{a\}$ and, since all the constant intervals have the cardinal of the continuum, we have $\mathbf{H}([0, 1]_*) = \mathbf{IS}([0, 1]_*) \cup \mathbf{IS}([0, 1]_*/F^a)$. Actually, the algebra $[0, 1]_*/F^a$ can also be seen as a standard WNM-chain since it is clearly isomorphic to a chain over $[0, 1]$. The reader can see an example of such situation in Figure 9.4.

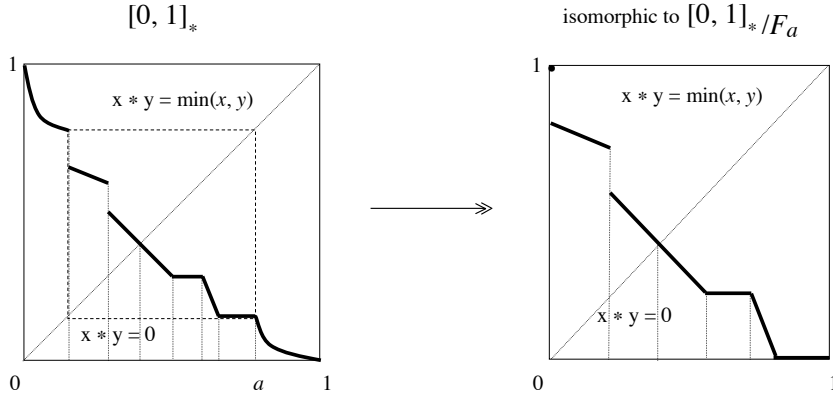


Figure 9.4: A WNM t-norm with a finite partition such that $I_1 = \{1\}$ (left) and its corresponding t-norm on the quotient algebra $[0, 1]_*/F_a$ (right).

Lemma 9.19. *Let \mathbb{K} be a class of WNM-chains closed under subalgebras. We have: $\mathbf{H}(\mathbb{K})_{fin} = \mathbf{H}(\mathbb{K}_{fin})$.*

Proof: One inclusion is trivial. As for the other one, take $\mathcal{A} \in \mathbf{H}(\mathbb{K})_{fin}$, then \mathcal{A} is a finite chain and it is isomorphic to \mathcal{B}/F for some $\mathcal{B} \in \mathbb{K}$ and some filter F of \mathcal{B} . The subalgebra of \mathcal{B} generated by $B \setminus (F \cup \overline{F})$ is in \mathbb{K} , thus $\mathcal{A} \in \mathbf{H}(\mathbb{K}_{fin})$. \square

Lemma 9.20. *Let \mathcal{A} be a WNM-chain. Then $\mathbf{ISP}_U(\mathcal{A})_{fin} = \mathbf{IS}(\mathcal{A})_{fin}$.*

Proof: One direction is obvious. Due to the local finiteness of $\mathbb{W}\mathbb{N}\mathbb{M}$, to prove the other one is equivalent to prove that $\mathbf{ISP}_U(\mathcal{A})$ is partially embeddable into $\mathbf{IS}(\mathcal{A})_{fin}$, which is equivalent by Proposition 2.8 to $\mathbf{ISP}_U(\mathcal{A}) \subseteq \mathbf{ISP}_U(\mathbf{IS}(\mathcal{A})_{fin})$; finally the last inclusion is true because $\mathcal{A} \in \mathbf{ISP}_U(\mathbf{IS}(\mathcal{A})_{fin})$. \square

Proposition 9.21. *Let \mathcal{A} be a WNM-chain. Then $\mathbf{HSP}_U(\mathcal{A})_{fin} = \mathbf{IS}(\mathcal{A})_{fin} \cup \mathbf{IS}(\{\mathcal{A}/F^a : a \in N(\mathcal{A}) \cap A_+ \text{ and } I_a \neq \{a\}\})_{fin}$.*

Proof: $\mathbf{HSP}_U(\mathcal{A})_{fin} = \mathbf{H}(\mathbf{ISP}_U(\mathcal{A}))_{fin} = [\text{by Lemma 9.19}] \mathbf{H}(\mathbf{ISP}_U(\mathcal{A})_{fin}) = [\text{by Lemma 9.20}] \mathbf{H}(\mathbf{IS}(\mathcal{A})_{fin}) = \mathbf{H}(\mathbf{IS}(\mathcal{A}))_{fin} = \mathbf{HS}(\mathcal{A})_{fin} = \mathbf{SH}(\mathcal{A})_{fin} = \mathbf{SIS}(\{\mathcal{A}/F^a : a = \bar{1}^{\mathcal{A}} \text{ or } a \in N(\mathcal{A}) \cap A_+ \text{ and } I_a \neq \{a\}\})_{fin} = \mathbf{IS}(\{\mathcal{A}/F^a : a = \bar{1}^{\mathcal{A}} \text{ or } a \in N(\mathcal{A}) \cap A_+ \text{ and } I_a \neq \{a\}\})_{fin}$. \square

Corollary 9.22. *Let \mathcal{A} be a WNM-chain such that it has the maximum positive involutive element a with $I_a \neq \{a\}$, and for any other $b \in N(\mathcal{A}) \cap A_+$, $|I_b| \leq |I_a|$. Then, $\mathbf{HSP}_U(\mathcal{A})_{fin} = \mathbf{IS}(\mathcal{A})_{fin} \cup \mathbf{IS}(\mathcal{A}/F^a)_{fin}$.*

The description of the classes $\mathbf{HSP}_U(\cdot)_{fin}$ leads to the following criterion to compare varieties generated by a finite family of chains.

Theorem 9.23. *Let $n, m \geq 1$ be natural numbers and let $\mathcal{A}_1, \dots, \mathcal{A}_n$ and $\mathcal{B}_1, \dots, \mathcal{B}_m$ be WNM-chains such that for every i there exists $a_i \in \mathcal{A}_i$ and $b_i \in \mathcal{B}_i$, positive involutive elements satisfying the conditions of the previous corollary. The following are equivalent:*

- (i) $\mathbf{V}(\mathcal{A}_1, \dots, \mathcal{A}_n) \subseteq \mathbf{V}(\mathcal{B}_1, \dots, \mathcal{B}_m)$
- (ii) $\mathbf{IS}(\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{A}_1/F^{a_1}, \dots, \mathcal{A}_n/F^{a_n})_{fin} \subseteq \mathbf{IS}(\mathcal{B}_1, \dots, \mathcal{B}_m, \mathcal{B}_1/F^{b_1}, \dots, \mathcal{B}_m/F^{b_m})_{fin}$.
- (iii) 1. For every $i \in \{1, \dots, n\}$, there is $j \in \{1, \dots, m\}$ such that $\mathbf{IS}(\mathcal{A}_i)_{fin} \subseteq \mathbf{IS}(\mathcal{B}_j)_{fin}$ or $\mathbf{IS}(\mathcal{A}_i)_{fin} \subseteq \mathbf{IS}(\mathcal{B}_j/F^{b_j})_{fin}$, and
2. for every $i \in \{1, \dots, n\}$, there is $k \in \{1, \dots, m\}$ such that $\mathbf{IS}(\mathcal{A}_i/F^{a_i})_{fin} \subseteq \mathbf{IS}(\mathcal{B}_k)_{fin}$ or $\mathbf{IS}(\mathcal{A}_i/F^{a_i})_{fin} \subseteq \mathbf{IS}(\mathcal{B}_k/F^{b_k})_{fin}$.

Proof: First observe that $\mathbf{V}(\mathcal{A}_1, \dots, \mathcal{A}_n) \subseteq \mathbf{V}(\mathcal{B}_1, \dots, \mathcal{B}_m)$ if, and only if, $\mathbf{V}(\mathcal{A}_1, \dots, \mathcal{A}_n)_{FSI} \subseteq \mathbf{V}(\mathcal{B}_1, \dots, \mathcal{B}_m)_{FSI}$. By Jónsson's Lemma and being $\mathbb{W}\mathbb{N}\mathbb{M}$ locally finite, this is equivalent to $\mathbf{HSP}_U(\mathcal{A}_1, \dots, \mathcal{A}_n)_{fin} \subseteq \mathbf{HSP}_U(\mathcal{B}_1, \dots, \mathcal{B}_m)_{fin}$. By the previous corollary, this is equivalent to: $\mathbf{IS}(\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{A}_1/F^{a_1}, \dots, \mathcal{A}_n/F^{a_n})_{fin} \subseteq \mathbf{IS}(\mathcal{B}_1, \dots, \mathcal{B}_m, \mathcal{B}_1/F^{b_1}, \dots, \mathcal{B}_m/F^{b_m})_{fin}$. Therefore, we have proved (i) \Leftrightarrow (ii). (iii) \Rightarrow (ii) is trivial.

(ii) \Rightarrow (iii): Suppose that (iii) does not hold. Then, for instance, there exists $i \in \{1, \dots, n\}$ such that for every $j \in \{1, \dots, m\}$, $\mathbf{IS}(\mathcal{A}_i)_{fin} \not\subseteq \mathbf{IS}(\mathcal{B}_j)_{fin}$ and $\mathbf{IS}(\mathcal{A}_i)_{fin} \not\subseteq \mathbf{IS}(\mathcal{B}_j/F^{b_j})_{fin}$. Therefore, there exist $\mathcal{C}_1, \dots, \mathcal{C}_m, \mathcal{D}_1, \dots, \mathcal{D}_m \in \mathbf{IS}(\mathcal{A}_i)_{fin}$ such that for every j , \mathcal{C}_j is not embeddable in \mathcal{B}_j and \mathcal{D}_j is not embeddable in \mathcal{B}_j/F^{b_j} . Consider the subalgebra $\mathcal{C} \subseteq \mathcal{A}_i$ generated by $\mathcal{C}_1 \cup \dots \cup \mathcal{C}_m \cup \mathcal{D}_1 \cup \dots \cup \mathcal{D}_m$. Then, \mathcal{C} is finite and it cannot belong to $\mathbf{IS}(\mathcal{B}_1, \dots, \mathcal{B}_m, \mathcal{B}_1/F^{b_1}, \dots, \mathcal{B}_m/F^{b_m})_{fin}$, so (ii) does not hold. \square

Therefore, finite WNM-chains will play a central role in the task of classifying varieties of WNM-algebras. Given a WNM-chain \mathcal{A} , the negation in \mathcal{A} only depends on the negation in $\mathcal{N}(\mathcal{A})$, due to the properties of Lemma 9.8. Therefore, every WNM-chain is characterized by the NM-subalgebra defined by its involutive elements and by the number of non-involutive elements in their associated intervals.

As in the case of NM-chains, some canonical representatives could be defined for the finite chains. Given $n \geq 1$, $l_1, \dots, l_n \geq 0$, $\mathcal{A}_{l_1, \dots, l_n}^n$ will denote the WNM-chain that has n involutive elements and l_i non-involutive elements in the constant interval of the $(i+1)$ -th involutive element. It is clear that these chains generate pairwise different varieties. We can see an example in Figure 9.5.

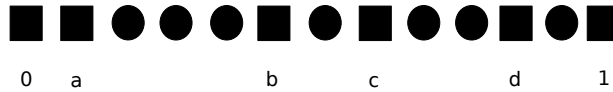


Figure 9.5: Example of a canonical finite WNM-chain, $\mathcal{A}_{0,3,1,2,1}^6$. Squares represent involutive elements, while circles represent the non-involutive ones. b, c, d and 1 have some associated non-involutive elements, while a (and, of course, 0) has a trivial associated interval.

9.3 Generic WNM-chains

In this section we will study the WNM-chains that generate the variety \mathbf{WNM} , i.e. the generic chains.

Definition 9.24. *Let \mathcal{A} be a WNM-chain. \mathcal{A} is generic if, and only if, $\mathbf{V}(\mathcal{A}) = \mathbf{WNM}$.*

They can be characterized by using Proposition 9.21 in the following way.

Theorem 9.25. *Let \mathcal{A} be a WNM-chain. The following are equivalent:*

- (1) \mathcal{A} is generic.
- (2) For every $\varphi \in \text{Fm}_{\mathcal{L}}$, $\mathcal{A} \models \varphi \approx \bar{1}$ if, and only if, $\vdash_{\text{WNM}} \varphi$.
- (3) For every finite WNM-chain \mathcal{C} , either \mathcal{C} is embeddable in \mathcal{A} or there is $a \in N(\mathcal{A}) \cap A_+$ such that $I_a \neq \{a\}$ and \mathcal{C} is embeddable in \mathcal{A}/F^a .

Some chains satisfy a condition stronger than (3), namely all finite chains are embeddable into them. This situation is characterized in the next proposition.

Proposition 9.26. *Let \mathcal{A} be a WNM-chain. Then, all finite WNM-chains are embeddable into \mathcal{A} if, and only if, it satisfies the following conditions:*

1. The set I_1 is infinite,
2. \mathcal{A} has a negation fixpoint f such that the set I_f is infinite, and
3. Either there is an increasing sequence $\langle a_n : n \in \omega \rangle$ of involutive elements in A_- whose limit is not f such that for every $n, k \geq 1$ there is $m \geq n$ such that the sets I_{a_m} and $I_{\neg a_m}$ have both more than k elements, or there is an increasing sequence $\langle a_n : n \in \omega \rangle$ of involutive elements in A_+ whose limit is not 1 such that for every $n, k \geq 1$ there is $m \geq n$ such that the sets I_{a_m} and $I_{\neg a_m}$ have both more than k elements.

Proof: If \mathcal{A} satisfies the conditions, it is obvious that every finite WNM-chain is embeddable into \mathcal{A} . In order to prove that the conditions are also necessary suppose that \mathcal{A} satisfies the first and the second condition but not the third (if the first or the second condition fail, then it is easy to produce a finite chain that it is not embeddable into \mathcal{A}). Consider the set $X = \{a \in A_- : a \text{ is involutive and } |I_a|, |I_{\neg a}| \geq \omega\}$. This set must be finite (otherwise \mathcal{A} would satisfy the third condition); suppose that X has m elements. For each involutive element $a \in A_-$, we define $r(a) := \min\{|I_a|, |I_{\neg a}|\}$. If $\{n(a) : a \in A_- \setminus X, a = \neg \neg a\}$ is unbounded, we produce a sequence by choosing $a_k \in \{a \in A_- \setminus X : a = \neg \neg a \text{ and } r(a) = k\}$ for every $k \in \omega$ such that $\{a \in A_- \setminus X : a = \neg \neg a \text{ and } r(a) = k\} \neq \emptyset$. But then we would have a sequence satisfying the third condition, contradicting our assumption. Hence, there is an upper bound k of $\{r(a) : a \in A_- \setminus X, a = \neg \neg a\}$. Then, it is clear that the finite chain $\mathcal{A}_{k+1, k+1, \dots, k+1}^{2m+4}$ is not embeddable into \mathcal{A} . \square

Figure 9.6 shows an example of a generic WNM-chain defined by a WNM-t-norm satisfying this stronger condition.

Furthermore, we obtain the following characterization of generic standard WNM-chains.

Theorem 9.27. *Let \mathcal{A} be a standard WNM-chain. Then, \mathcal{A} is generic if, and only if, it satisfies the following conditions:*

1. \mathcal{A} has a negation fixpoint f such that the set I_f is infinite, and

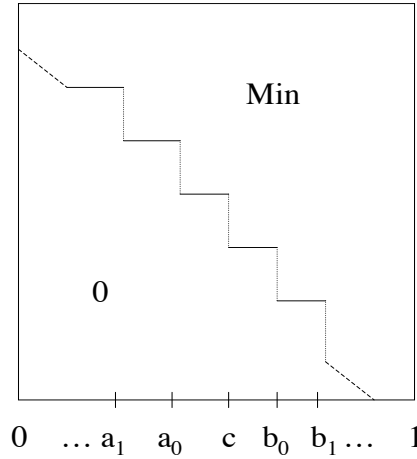


Figure 9.6: Example of a generic chain \mathcal{A} defined by a WNM-t-norm over the real unit interval $[0, 1]$. It has a decreasing sequence $\langle a_n : n \in \omega \rangle$ of involutive elements in the negative part with a non-trivial associated interval, an increasing sequence $\langle b_n : n \in \omega \rangle$ of involutive elements in the positive part with a non-trivial associated interval, a fixpoint c with $I_c \neq \{c\}$, and $I_1 \neq \{1\}$.

2. *Either there is an increasing sequence $\langle a_n : n \in \omega \rangle$ of involutive elements in A_- whose limit is not f such that for every $n, k \geq 1$ there is $m \geq n$ such that the sets I_{a_m} and $I_{\neg a_m}$ have both more than k elements, or there is an increasing sequence $\langle a_n : n \in \omega \rangle$ of involutive elements in A_+ whose limit is not 1 such that for every $n, k \geq 1$ there is $m \geq n$ such that the sets I_{a_m} and $I_{\neg a_m}$ have both more than k elements.*

Proof: Assume that \mathcal{A} is generic. If there is a maximum constant interval I_a (with possibly $a = 1$), then every finite WNM-chain is embeddable in \mathcal{A}/F^a . Hence, by Proposition 9.26, \mathcal{A}/F^a satisfies the conditions, so also \mathcal{A} satisfies them. Suppose now that the maximum constant interval does not exist. Since all finite chains are embeddable in \mathcal{A}/F^a for some suitable a , it is clear that \mathcal{A} has a negation fixpoint f and the set I_f is infinite. If it would not satisfy the other condition, then the set $\{a \in A_- : I_a \text{ and } I_{\neg a} \text{ are infinite}\}$ would be finite, and then it would be possible to find a finite chain which we could not embed

in any quotient of \mathcal{A} ; a contradiction.

Conversely, suppose that \mathcal{A} satisfies the two conditions. Then it is clear that every finite WNM-chain is embeddable in some quotient of \mathcal{A} . \square

9.4 T-norm based axiomatic extensions of the Weak Nilpotent Minimum logic and their standard completeness properties

In this section we focus on varieties generated by t-norm-algebras, i.e. standard WNM-chains.

Lemma 9.28. *Let $[0, 1]_*$ be a standard WNM-chain. If $I_1 \neq \{1\}$, then $\mathbf{HSP}_U([0, 1]_*)_{fin} = \mathbf{IS}([0, 1]_*)_{fin}$*

Proof: Just apply the Corollary 9.22 with $a = 1$. \square

This gives the following criterion to compare varieties generated by standard WNM-chains such that $I_1 \neq \{1\}$.

Corollary 9.29. *Let \mathcal{A} and \mathcal{B} be standard WNM-chains such that $I_1^{\mathcal{A}} \neq \{1\}$ and $I_1^{\mathcal{B}} \neq \{1\}$. Then the following are equivalent:*

- $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{B})$
- $\mathbf{IS}(\mathcal{A})_{fin} \subseteq \mathbf{IS}(\mathcal{B})_{fin}$.

We can obtain similar results for t-norms satisfying the FPP.

Lemma 9.30. *Let $[0, 1]_*$ be a standard WNM-chain. If $*$ \in **WNM-fin** and I_a is the maximum constant interval, then $\mathbf{HSP}_U([0, 1]_*)_{fin} = \mathbf{IS}([0, 1]_*)_{fin} \cup \mathbf{IS}([0, 1]_*/F^a)_{fin}$.*

Proof: By Corollary 9.22. \square

Corollary 9.31. *Let \mathcal{A} and \mathcal{B} be standard WNM-chains with finite partition such that $I_a^{\mathcal{A}}$ and $I_b^{\mathcal{B}}$ are their maximum constants intervals respectively. Then the following are equivalent:*

- $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{B})$
- $\mathbf{IS}(\mathcal{A})_{fin} \cup \mathbf{IS}(\mathcal{A}/F^a)_{fin} \subseteq \mathbf{IS}(\mathcal{B})_{fin} \cup \mathbf{IS}(\mathcal{B}/F^b)_{fin}$.

Notice that corollaries 9.29 and 9.31 give a classification of varieties generated by a standard WNM-chain (when the chains have $I_1 \neq \{1\}$ or satisfy the FPP). Indeed, if \mathcal{A} and \mathcal{B} are standard WNM-chains under these conditions, the inclusion of the set of finite subalgebras of \mathcal{A} into the of finite subalgebras of \mathcal{B} is easy to compute, since the possible finite subalgebras only depend on

the partitions of \mathcal{A} and \mathcal{B} . The results can be easily generalized to varieties generated by a family of standard WNM-chains.

Remark: It is easy to see that if $*$ \in **WNM-fin**, then all the chains in the variety $\mathbf{V}([0, 1]_*)$ enjoy the FPP. Indeed, we can equationally express the maximum number of constant intervals that these chains can have in their partitions. Suppose, for instance, that $[0, 1]_*$ is the standard WNM-chain depicted in Figure 9.3 and consider the following equations:

$$(E_1) \quad \neg\neg n(x) \rightarrow n(x) \approx \bar{1}$$

$$(E_2) \quad (\neg x_0 \leftrightarrow \neg x_0) \vee (\neg x_1 \leftrightarrow \neg x_1) \vee (\neg x_2 \leftrightarrow \neg x_2) \vee (\neg p(x_0) \rightarrow p(x_0)) \vee (\neg p(x_1) \rightarrow p(x_1)) \vee (\neg p(x_2) \rightarrow p(x_2)) \vee (\neg p(x_0) \rightarrow \neg p(x_1)) \vee (\neg p(x_1) \rightarrow \neg p(x_2)) \approx \bar{1}$$

It is not difficult to check that any WNM-chain satisfying (E_1) has only involutive elements in the negative part, and any WNM-chain satisfying (E_2) has at most 2 constant intervals in the positive part. Since these equations are valid in $[0, 1]_*$, they are also valid in all the chains in $\mathbf{V}([0, 1]_*)$, and hence all of them enjoy the FPP.

Given any standard WNM-chain $[0, 1]_*$ it is obvious that the logic L_* , i.e. the logic corresponding to the variety $\mathbf{V}([0, 1]_*)$, enjoys the canonical SC with respect to $[0, 1]_*$. Now we will study in which cases this standard completeness result can be improved. We start with t-norms satisfying the FPP.

Proposition 9.32. *Let $*$ \in **WNM-fin**, let I_a be its maximum constant interval (with possibly $a = 1$) and let \mathcal{A} be a countable L_* -chain. Then:*

- *If $I_{\bar{1}\mathcal{A}}^{\mathcal{A}} = \{\bar{1}^{\mathcal{A}}\}$, then there exists an embedding from \mathcal{A} into $[0, 1]_*$.*
- *If $I_{\bar{1}\mathcal{A}}^{\mathcal{A}} \neq \{\bar{1}^{\mathcal{A}}\}$, then there exists an embedding from \mathcal{A} into $[0, 1]_*/F^a$.*

Proof: We are assuming that $[0, 1]_*$ has a finite partition. Let r and s be the number of intervals in the negative part, and respectively in the positive part, of $[0, 1]_*$. Suppose that $I_{\bar{1}\mathcal{A}}^{\mathcal{A}} = \{\bar{1}^{\mathcal{A}}\}$. By the last remark we know that the number of intervals in the negative part (resp. in the positive part) of \mathcal{A} is at most r (resp. s). Take a finite WNM-subchain \mathcal{B} satisfying the following construction rules:

1. Every unitary interval belonging to the partition of \mathcal{A} is in B .
2. For every non-unitary constant interval of the partition of \mathcal{A} , one interior element of this interval and its upper bound belong to B .
3. For every involutive non-unitary interval in the negative part of the partition of \mathcal{A} , two different elements and their negations belong to B .

It is clear see that such a chain exists and it is a finite WNM-chain, subalgebra of \mathcal{A} , with the same number of intervals in the partition. By Lemma 9.30, there is an embedding $g : \mathcal{B} \hookrightarrow [0, 1]_*$. Observe now that two different non-unitary intervals of the partition of \mathcal{B} must be embedded into two different intervals of the partition of $[0, 1]_*$ and also that as subalgebra two different intervals of the partition of \mathcal{B} are contained in two different intervals of the partition of \mathcal{A} . Thus, remembering that the non-unitary intervals of \mathcal{A} are countable and the ones in $[0, 1]_*$ are continuous, and using that \mathcal{A} and \mathcal{B} have the same partition, we can define an embedding $f : \mathcal{A} \hookrightarrow [0, 1]_*$. If $I_{1\mathcal{A}}^{\mathcal{A}} \neq \{\bar{1}^{\mathcal{A}}\}$, the proof is analogous. \square

Corollary 9.33. *Let $*$ \in WNM-fin and let I_a be its maximum constant interval. Then:*

- *If $a = 1$, then the logic L_* has the canonical SSC with respect to $[0, 1]_*$.*
- *If $a \neq 1$, then the logic L_* has the SSC with respect to $\{[0, 1]_*, [0, 1]_*/F^a\}$.*

Now we turn to t-norms with an infinite partition.

Proposition 9.34. *Given $*$ \in WNM \ WNM-fin, an L_* -chain \mathcal{C} and a finite partial subalgebra $\mathcal{B} \subseteq_p \mathcal{C}$, we have:*

- *If $I_1^* \neq \{1\}$, then \mathcal{B} is partially embeddable in $[0, 1]_*$.*
- *If $I_1^* = \{1\}$, then \mathcal{B} is partially embeddable in $[0, 1]_*$ or there is a positive involutive element $a \in [0, 1]$ with $I_a^* \neq \{a\}$ such that \mathcal{B} is partially embeddable in $[0, 1]_*/F^a$.*

Proof: Since WNM is locally finite, the subalgebra of \mathcal{C} generated by \mathcal{B} is also finite. Then, Proposition 9.21 gives the result. \square

Corollary 9.35. *Given $*$ \in WNM \ WNM-fin, we have:*

- *If $I_1^* \neq \{1\}$, then the logic L_* has the canonical FSSC with respect to $[0, 1]_*$.*
- *If $I_1^* = \{1\}$, then the logic L_* has the FSSC with respect to $\{[0, 1]_*\} \cup \{[0, 1]_*/F^a : a \text{ is positive, involutive and } I_a \neq \{a\}\}$.*

Although in some cases the SSC holds for logics of WNM-t-norms with an infinite partition (for instance, when $[0, 1]_*$ is a generic WNM-chain), it is false in general as the following examples show.

Example 4. *Let $[0, 1]_*$ a standard WNM-chain with an infinite partition such that the number of positive constant intervals is finite, say $I_{a_1}^*, \dots, I_{a_n}^*$. Assume that $I_1^* \neq \{1\}$. For every i , let X_i be the set of these discontinuity points of the negation between $I_{a_i}^*$ and $I_{a_{i+1}}^*$ and let Y_{i1} the set of accumulation points of X_i which are a limit of an increasing sequence of elements of X_i , and let Y_{i2} the set of accumulation points of X_i which are a limit of a decreasing sequence of elements of X_i . Take $a \in I_1^* \setminus \{1\}$ and let \mathcal{A} be the countable subalgebra of $[0, 1]_*$ generated by the rational numbers in $[0, a]$. It is clear that \mathcal{A} is subdirectly irreducible. Assume that there is i such that X_i is infinite and Y_{i1} or Y_{i2} is finite. Then:*

1. If Y_{i1} is finite, we can produce a new countable WNM-chain \mathcal{B} by adding to \mathcal{A} a new accumulation point to Y_{i1} .
2. If Y_{i2} is finite, we can produce a new countable WNM-chain \mathcal{B} by adding to \mathcal{A} a new accumulation point to Y_{i2} .

In both cases, $\mathcal{B} \in \mathbf{V}([0, 1]_*)$, since every finite subalgebra of \mathcal{B} is embeddable in $[0, 1]_*$, but clearly \mathcal{B} is not embeddable in $[0, 1]_*$. Therefore, \mathbf{L}_* has not the SSC.

An analogous reasoning is possible when the number of negative constant intervals is finite.

Example 5. Let $[0, 1]_*$ a standard WNM-chain with an infinite partition such that 1 is the only accumulation point of positive constant intervals. Consider the formula $\varphi_{>}(x, y) := (y \rightarrow x) \wedge ((x \rightarrow y) \rightarrow y)$. The following claim is easy to check.

Claim: For every WNM-chain \mathcal{A} and every $a, b \in A$ we have:

$$\varphi_{>}^{\mathcal{A}}(a, b) = \bar{1}^{\mathcal{A}} \text{ iff } a > b \text{ or } a = b = \bar{1}^{\mathcal{A}}.$$

Using this formula, we can formulate an infinite semantical derivation, $\{\varphi_{>}(\neg\neg p(x_{i+1}), \neg\neg p(x_i)) : i \geq 1\} \cup \{\varphi_{>}(\neg\neg p(x_i), p(x_i)) : i \geq 1\} \cup \{\varphi_{>}(\neg\neg p(x_0), \neg\neg p(x_i)) : i \geq 1\} \models_{[0,1]_*} \neg\neg p(x_0) \vee \neg\neg p(x_1)$, but it is not valid if we consider only a finite subset of the premisses, so \mathbf{L}_* has not the SSC.

An analogous reasoning is possible when the only accumulation point of positive constant intervals is the infimum of the positive elements.

9.5 Axiomatization of some t-norm based extensions of the Weak Nilpotent Minimum logic

In this section we give finite equational bases for some varieties generated by standard WNM-chains, or equivalently finite axiomatizations for some t-norm based extensions of WNM. Since every variety is generated by its finite chains, the equational base essentially has to describe these finite chains. More precisely:

Lemma 9.36. Given a WNM-chain \mathcal{A} , the following statements are equivalent:

1. The variety $\mathbf{V}(\mathcal{A})$ is axiomatized by the equations $\Sigma \subseteq Eq_{\mathcal{L}}$.
2. For every finite WNM-chain \mathcal{C} , $\mathcal{C} \in \mathbf{HSP}_U(\mathcal{A})$ iff $\mathcal{C} \models \Sigma$.
3. For every finite WNM-chain \mathcal{C} , $\mathcal{C} \in \mathbf{IS}(\mathcal{A}) \cup \mathbf{IS}(\{\mathcal{A}/F^a : a \in N(\mathcal{A}) \cap A_+ \text{ and } I_a \neq \{a\}\})$ iff $\mathcal{C} \models \Sigma$.

We will focus on the last condition, which is the most descriptive.

First, we consider some easy observations on the equations in the language of MTL.

Lemma 9.37. Let \mathcal{A} be an MTL-algebra, let $\varphi \approx \psi \in Eq_{\mathcal{L}}$ be an equation and $\Sigma = \{\varphi_i \approx \psi_i : i < n\} \subseteq Eq_{\mathcal{L}}$ be a finite set of equations. Then:

1. $\mathcal{A} \models \varphi \approx \psi$ if, and only if, $\mathcal{A} \models \varphi \leftrightarrow \psi \approx \bar{1}$, and
2. $\mathcal{A} \models \Sigma$ if, and only if, $\mathcal{A} \models (\varphi_0 \leftrightarrow \psi_0) \& \dots \& (\varphi_{n-1} \leftrightarrow \psi_{n-1}) \approx \bar{1}$.

Therefore, every finite equational base can be reduced to one single equation whose second member is the constant for the neutral element of the monoid. Using this and the following lemma we can produce an equational base for the variety generated by a finite family of MTL-chains, whenever we have an equational base for the variety generated by each chain of the family.

Lemma 9.38. *Let $p_0(\bar{x}_0), \dots, p_n(\bar{x}_n) \in Fm_{\mathcal{L}}$, where $\bar{x}_0, \dots, \bar{x}_n$ denote pairwise disjoint sets of variables. Let \mathcal{A} be an MTL-chain. Then, $\mathcal{A} \models p_0(\bar{x}_0) \vee \dots \vee p_n(\bar{x}_n) \approx \bar{1}$ if, and only if, there exists $i \leq n$ such that $\mathcal{A} \models p_i(\bar{x}_i) \approx \bar{1}$.*

Corollary 9.39. *Let $\{\mathcal{C}_1, \dots, \mathcal{C}_n\}$ be a finite set of WNM-chains. Suppose that for each $i \in \{1, \dots, n\}$, $p_i \approx \bar{1}$ is an equation axiomatizing $\mathbf{V}(\mathcal{C}_i)$, in such a way that the sets of variables of these equations are pairwise disjoint. Then, the equation $p_1 \vee \dots \vee p_n \approx \bar{1}$ axiomatizes the variety $\mathbf{V}(\{\mathcal{C}_1, \dots, \mathcal{C}_n\})$.*

In the following we provide some examples of t-norm based axiomatic extensions of WNM for which we are able to give efectively a finite axiomatization.

Examples: Let $*$ be a WNM-t-norm, let $[0, 1]_*$ be its corresponding standard WNM-algebra and let L_* be the axiomatic extension of WNM corresponding to the variety $\mathbf{V}([0, 1]_*)$. Our aim is to find a set of axiom schemata such that, added to the Hilbert-style system for WNM, give a calculus for L_* (or equivalently, to find a set of equations such that, added to the equational base for WNM, give an equational base for $\mathbf{V}([0, 1]_*)$).

1. If $[0, 1]_*$ is a generic WNM-t-norm (i.e. it satisfies the conditions of Theorem 9.27), then L_* is just WNM, and hence there is no need for additional axioms.
2. Suppose that $[0, 1]_*$ satisfies the following conditions:
 - The partition of $[0, 1]_*$ has no constant interval in the negative part.
 - $[0, 1]_*$ has a negation fixpoint.
 - The partition of $[0, 1]_*$ has infinitely many constant intervals in the positive part (i.e. $*$ \in $\mathbf{WNM} \setminus \mathbf{WNM}\text{-fin}$).

It is clear that for every finite WNM-chain \mathcal{C} , $\mathcal{C} \in \mathbf{IS}([0, 1]_*) \cup \mathbf{IS}(\{[0, 1]_*/F^a : a \in N[0, 1]_* \cap ([0, 1]_*)_+ \text{ and } I_a \neq \{a\}\})$ iff all the negative elements in \mathcal{C} are involutive. Therefore, the variety generated by $[0, 1]_*$ is axiomatized by the following equation:¹

$$\neg \neg n(x) \approx n(x)$$

¹Its associated logic has been already studied in [69] under with a different name, $\text{MTL}[D_\wedge]$, and a different axiomatization. In particular, the authors proved the SSC for this logic.

Notice that the symmetric situation (no constant intervals in the positive part, while infinitely many in the negative part) is axiomatized by:

$$\neg\neg p(x) \approx p(x)$$

Of course, the two equations together would give the variety generated by $[0, 1]_{\text{NM}}$, which can be axiomatized just by:

$$\neg\neg x \approx x$$

3. Suppose that $[0, 1]_*$ satisfies the following condition:

- There is a sequence, either increasing or decreasing, $\langle a_n : n \in \omega \rangle$ of involutive elements in A_- such that for every $n \geq 0$ there is $m \geq n$ such that the sets I_{a_m} and $I_{\neg a_m}$ are non-trivial.

i.e. just the second condition required for generic standard WNM-chains in Theorem 9.27. On the one hand, it is clear that for every finite WNM-chain \mathcal{C} , $\mathcal{C} \in \mathbf{IS}([0, 1]_*) \cup \mathbf{IS}(\{[0, 1]_*/F^a : a \in N[0, 1]_* \cap ([0, 1]_*)_+ \text{ and } I_a \neq \{a\}\})$ iff \mathcal{C} has no negation fixpoint. On the other hand, from the results in Chapter 6, we know that a WNM-chain is perfect iff it has no negation fixpoint. Therefore, the equation for perfect MTL-chains will be enough to obtain an equational base for the variety we are considering now:

$$(\neg(\neg x)^2)^2 \approx \neg(\neg x^2)^2$$

4. Take $* \in \mathbf{WNM-fin}$ such that the partition of $[0, 1]_*$ has no involutive intervals. Let r and s be respectively the number of constant intervals in the negative and in the positive part. Then, due to the symmetry properties of negation functions, we obtain that:

- If $[0, 1]_*$ has no negation fixpoint, then $s = r + 1$.
- If $[0, 1]_*$ has negation fixpoint, then $s = r$.

Observe that these chains (we can see two examples in Figure 9.7) have a finite number of involutive elements: 0 and the right extreme of each constant interval:

- If $[0, 1]_*$ has no negation fixpoint, then it has $2r + 2$ involutive elements.
- If $[0, 1]_*$ has negation fixpoint, then it has $2r + 1$ involutive elements.

Therefore, in order to axiomatize this kind of varieties we only need an equation giving an upper bound to the number of involutive elements. It is easy to check that a WNM-chain \mathcal{A} has at most k involutive elements if, and only if, the following equation is valid in \mathcal{A} :

$$\bigvee_{i < k} (\neg x_i \rightarrow \neg x_{i+1}) \approx \bar{1}$$

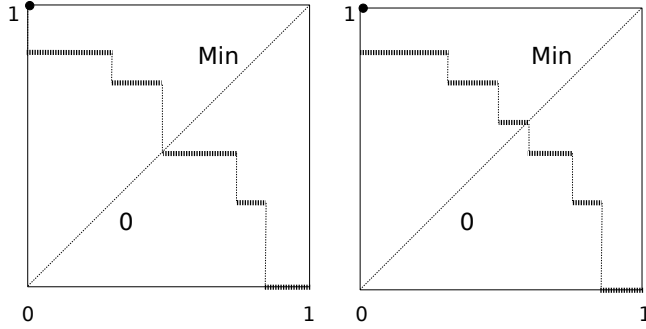


Figure 9.7: Two examples of standard WNM-chains satisfying the FPP with no involutive intervals. The chain on the left hand side has no negation point, while the chain on the right hand side has it.

For instance, to axiomatize the varieties corresponding to the chains in Figure 9.7, we would take the equation with $k = 6$ (for the chain on the left hand side) and the equation with $k = 7$ (for the chain on the right hand side).

5. Finally, assume that $* \in \mathbf{WNM}\text{-fin}$ and the partition of $[0, 1]_*$ has some involutive intervals. We have not found an equational for every WNM-t-norm under these conditions. However, we will illustrate with some example how it could be done when the partition has a small number of intervals. For instance, suppose that $*$ is the t-norm depicted in Figure 9.2. In this case the equational base only needs to force the chains to have no constant intervals in the negative part and at most 2 in the positive part. Thus we take the equations:

$$\neg\neg n(x) \approx n(x)$$

and

$$\bigvee_{i < 3} (\neg x_i \leftrightarrow \neg\neg x_i) \vee \bigvee_{i < 3} (\neg\neg p(x_i) \rightarrow p(x_i)) \vee \bigvee_{i < 2} (\neg\neg p(x_i) \rightarrow \neg\neg p(x_{i+1})) \approx \bar{1}$$

Consider now the chain in Figure 9.8 where some more restrictions must be described in the equations.

In this case we take the following equational base:

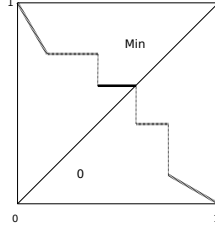


Figure 9.8: An example of a standard WNM-chain satisfying the FPP with involutive intervals.

$$\bigvee_{i < 3} (\neg \neg n(x_i) \rightarrow n(x_i)) \vee \bigvee_{i < 2} (\neg \neg n(x_i) \rightarrow \neg \neg n(x_{i+1})) \approx \bar{1}$$

(there are at most two constant intervals in the negative part)

$$\bigvee_{i < 2} (\neg x_i \leftrightarrow \neg \neg x_i) \vee \bigvee_{i < 2} (\neg \neg p(x_i) \rightarrow p(x_i)) \vee (\neg \neg p(x_0) \rightarrow \neg \neg p(x_1)) \approx \bar{1}$$

(there is at most one constant interval in the positive part)

$$\bigvee_{i < 2} (\neg \neg n(x_i) \rightarrow n(x_i)) \vee \bigvee_{i < 2} (\neg \neg n(x_i) \rightarrow \neg \neg n(x_{i+1})) \vee (\neg \neg n(y) \rightarrow \neg \neg n(x_1)) \vee (\neg \neg n(x_0) \rightarrow \neg \neg n(y)) \approx \bar{1}$$

(if there are two constant intervals in the negative part, then there are no involutive elements between them)

$$\bigvee_{i < 2} (\neg \neg n(x_i) \rightarrow n(x_i)) \vee \bigvee_{i < 2} (\neg \neg n(x_i) \rightarrow \neg \neg n(x_{i+1})) \vee (\neg \neg n(x_0) \leftrightarrow \neg \neg n(x_0)) \approx \bar{1}$$

(if there are two constant intervals in the negative part, then the right extreme of the second one is the negation fixpoint)

$$(\neg \neg n(x_0) \rightarrow n(x_0)) \vee (\neg \neg n(y_0) \rightarrow \neg \neg n(y_1)) \vee (\neg \neg n(y_1) \rightarrow \neg \neg n(x_0)) \approx \bar{1}$$

(if there is a constant interval in the negative part, then there is at most one negative involutive element above it)

$$(\neg x_0 \leftrightarrow \neg \neg x_0) \vee (\neg \neg p(x_0) \rightarrow p(x_0)) \vee (\neg y_0 \leftrightarrow \neg \neg y_0) \vee (\neg \neg p(x_0) \rightarrow \neg \neg p(y_0)) \approx \bar{1}$$

(if there is a constant interval in the positive part, then there are no positive involutive elements below it)

9.6 Conclusions

In this chapter we have studied a particular variety of MTL-algebras, WNM , which is a proper subvariety of the intersection of the varieties $\mathcal{C}_3\text{MTL}$ and $\mathcal{BP}_0^{+\omega}$, introduced in previous chapters. After presenting the description and axiomatization of the varieties formed by the involutive members obtained in [71], we have achieved the following new results:

- \mathbf{WNM} is a locally finite variety, so it has the FEP and the FMP and the corresponding logic is decidable. Obviously, these properties are inherited by all the subvarieties and axiomatic extensions, respectively.
- We have studied \mathbf{WNM} -t-norms. In particular, we have characterized the generic t-norms, we have given criteria to compare their generated varieties and we have studied their standard completeness properties.
- We have given equational bases for some varieties generated by a finite family of standard \mathbf{WNM} -chains.

Part II

Partial truth in triangular norm based logics

Chapter 10

Expansions with truth-constants

T-norm based fuzzy logics are basically logics of *comparative truth*. In fact, the residuum \Rightarrow of a (left-continuous) t-norm $*$ satisfies for every $x, y \in [0, 1]$ the condition $x \Rightarrow y = 1$ if, and only if, $x \leq y$. This means that a formula $\varphi \rightarrow \psi$ is a logical consequence of a theory if the truth-degree of φ is at most as high as the truth degree of ψ in any interpretation which is a model of the theory. In fact the logic of continuous t-norms as it is presented in Hájek's seminal book [79], only deals with valid formulae and deductions using 1 as the only designated truth-value. This line is followed by the majority of logical papers written from then in the setting of many-valued fuzzy logics.

But, in general, these systems do not exploit in depth neither the idea of comparative truth nor the potentiality of dealing with explicit partial truth that a many-valued logic setting offers. On the one hand, for instance, a logic which is based exclusively on the idea of comparative truth is the system L_{∞}^{\leq} (see [60]) where a deduction is valid if, and only if, the degree of truth of the premises is less or equal than the degree of truth of conclusion. The system developed there is based on Łukasiewicz infinitely-valued logic L but it could be defined over any t-norm based logic. Actually, since Gödel logic G is the only t-norm based logic enjoying the classical deduction-detachment theorem, it is the only case in which the usual G logic coincides with G_{∞}^{\leq} .

On the other hand, in some situations one might be also interested to explicitly represent and reason with partial degrees of truth. To do so, one convenient and elegant way is introducing truth-constants into the language. This approach actually goes back to Pavelka [128] who built a propositional many-valued logical system which turned out to be equivalent to the expansion of Łukasiewicz Logic L by adding into the language a truth-constant \bar{r} for each *real* $r \in [0, 1]$, together with a number of additional axioms. Although the resulting logic is not strongly complete with respect to the intended semantics defined by the Łukasiewicz t-norm, (like the original Łukasiewicz logic), Pavelka proved that his logic, denoted

here PL, is complete in a different sense. Namely, he defined the *truth-degree* of a formula φ in a theory T as $\|\varphi\|_T = \inf\{e(\varphi) \mid e \text{ is a PL-evaluation model of } T\}$, and the *provability degree* of φ in T as $|\varphi|_T = \sup\{r \mid T \vdash_{\text{PL}} \bar{r} \rightarrow \varphi\}$ and proved that these two degrees coincide. This kind of completeness is usually known as *Pavelka-style completeness*, and strongly relies on the continuity of Łukasiewicz truth functions. Novák extended Pavelka's approach to Łukasiewicz first-order logic [123]. Furthermore, Łukasiewicz logic extended with truth-constants has been extensively developed by Nývák and colleagues in the frame of the so-called *fuzzy logic with evaluated syntax* (see e.g. [124]).

Later, Hájek [79] showed that the logic PL could be significantly simplified while keeping the Pavelka-style completeness results. Indeed he showed that it is enough to extend the language only by a countable number of truth-constants, one constant \bar{r} for each *rational* in $r \in [0, 1]$, and by adding to the logic the two following additional axiom schemata, called book-keeping axioms:

$$\begin{aligned}\bar{r} \&\bar{s} &\leftrightarrow \bar{r} *_{\text{L}} \bar{s} \\ (\bar{r} \rightarrow \bar{s}) &\leftrightarrow \bar{r} \Rightarrow_{\text{L}} \bar{s}\end{aligned}$$

where $*_{\text{L}}$ and \Rightarrow_{L} are the Łukasiewicz t-norm and its residuum respectively. He called this new system Rational Pavelka Logic, RPL for short. Moreover, he proved that RPL is strongly complete (in the usual sense) for finite theories.

Similar *rational* expansions for other continuous t-norm based fuzzy logics can be analogously defined, but Pavelka-style completeness cannot be obtained since Łukasiewicz Logic is the only fuzzy logic whose truth-functions are a continuous t-norm and a continuous residuum.¹

However, several expansions with truth-constants of fuzzy logics different from Łukasiewicz have been studied, mainly related to the other two outstanding continuous t-norm based logics, namely Gödel and Product logic. We may cite [79] where an expansion of G_{Δ} (the expansion of Gödel Logic G with Baaz's projection connective Δ) with a finite number of rational truth-constants is studied, [54] where the authors define logical systems obtained by adding (rational) truth-constants to G_{\sim} (Gödel Logic with an involutive negation) and to Π (Product Logic) and Π_{\sim} (Product Logic with an involutive negation). In the case of the rational expansions of Π and Π_{\sim} an infinitary inference rule (from $\{\varphi \rightarrow \bar{r} : r \in \mathbb{Q} \cap (0, 1)\}$ infer $\varphi \rightarrow \bar{0}$) is introduced in order to obtain Pavelka-style completeness. Rational truth-constants have been also considered in some

¹An easy argument shows that for logics based on other continuous t-norms Pavelka-style completeness does not hold. Let L_* be the logic of a continuous t-norm $*$ (not isomorphic to Łukasiewicz t-norm) and its residuum \Rightarrow (as defined in [53]). Then it is known that the induced negation $\neg x = x \Rightarrow 0$ is not continuous in $x = 0$, i.e. $\sup\{\neg x \mid x > 0\} < \neg 0 = 1$.

Let p be a propositional variable and let $T = \{p \rightarrow \bar{r} \mid r > 0\}$. One can show that $\|p \rightarrow \bar{0}\|_T \neq |p \rightarrow \bar{0}|_T$. Indeed, $\|p \rightarrow \bar{0}\|_T = \inf\{e(p) \Rightarrow 0 \mid e(p) \leq r \text{ for all } r > 0\} = 0 \Rightarrow 0 = 1$, and we show that $|p \rightarrow \bar{0}|_T < 1$. For this, it is enough to prove that $T \not\vdash \bar{r}_0 \rightarrow (p \rightarrow \bar{0})$ for any $r_0 < 1$ such that $r_0 > \sup\{\neg x \mid x > 0\}$ (such an element exists because $*$ is not isomorphic to Łukasiewicz t-norm). Suppose not. In such a case, there would exist a finite theory $T_0 \subseteq T$ such that $T_0 \vdash \bar{r}_0 \rightarrow (p \rightarrow \bar{0})$. Then, by soundness, it should be $r_0 \leq \neg e(p)$ for any evaluation e such that $e(p) \leq s$, where $s = \min\{r \mid \bar{r} \rightarrow p \in T_0\}$, which is a contradiction (e.g. take $e(p) = s$).

stronger logics like in the logic $L\Pi_2^1$ [53], a logic that combines the connectives from both Łukasiewicz and Product logics plus the truth-constant $\overline{1/2}$, and in the logic PL [92], a logic which combines Łukasiewicz Logic connectives plus an additional conjunction, as well as in some closely related logics.

Following this line, Cintula gives in [37] a definition of what he calls *Pavelka-style extension* of a particular fuzzy logic. He considers the Pavelka-style extensions of the most popular fuzzy logics, and for each one of them he defines an axiomatic system with infinitary rules (to overcome discontinuities like in the case of Π explained above) which is proved to be Pavelka-style complete. Moreover he also considers the first-order versions of these extensions and provides necessary conditions for them to satisfy Pavelka-style completeness.

In this chapter, the approach based on traditional algebraic semantics will be considered in order to study completeness results (in the usual sense) for expansions of t-norm based logics with truth-constants. Indeed, as already mentioned, only the case of Łukasiewicz logic was known after [79]. Now we will provide a full description of completeness results for the expansions of logics of t-norms with a set of truth-constants $\{\bar{r} \mid r \in C\}$, for a suitable countable $C \subseteq [0, 1]$, when (i) the t-norm is either a finite ordinal sum of Łukasiewicz, Gödel and Product components or a WNM that has a finite partition and (ii) the set of truth-constants *covers* all the unit interval in the sense that each component (for continuous case) or interval of the partition (for the WNM case) contains at least one value of C in its interior.

All the results included in this chapter have been obtained in a long-term cooperation of the author with several researchers and they have been already published (or submitted for publication) in form of several papers. More precisely, the expansion of Gödel (and of some t-norm based logic related to the Nilpotent Minimum t-norm) with rational truth-constants and the expansion of Product logic with countable sets of truth-constants have been studied in [55] and in [132]. Later on, the basic cases of Łukasiewicz, Gödel and Product logics have been recently extended in [50] to the more general case of logics of continuous t-norms which are finite ordinal sums of the three basic components. Finally, in [56] some other completeness results corresponding to the expansions with truth-constants of logics of WNM t-norms with a finite partition have been added. In these papers, the issue of canonical standard completeness (that is, completeness with respect to the standard algebra where the truth-constants are interpreted as their own values) for these logics has been determined. Also, special attention has been paid to the fragment of formulae of the kind $\bar{r} \rightarrow \varphi$, where φ is a formula without additional truth-constants. Actually, this kind of formulae have been extensively considered in other frameworks for reasoning with partial degrees of truth, like in Novák's evaluated syntax formalism based on Łukasiewicz Logic (see e.g. [125]), in Gerla's framework of abstract fuzzy logics [70] or in fuzzy logic programming (see e.g. [138]).

10.1 Adding truth-constants

In this section we introduce the basic definitions and first general results regarding the expansions with truth-constants for those extensions of MTL which are the logic of a given left-continuous t-norm. In the following, for any left-continuous t-norm $*$, $[0, 1]_* = \langle [0, 1], *, \Rightarrow, \min, \max, 0, 1 \rangle$ is its corresponding standard MTL-chain and L_* will denote its corresponding axiomatic extension of MTL.

Definition 10.1 (logic $L_*(\mathcal{C})$). *Let $*$ be a left-continuous t-norm, and let $\mathcal{C} = \langle C, *, \Rightarrow, \min, \max, 0, 1 \rangle \subseteq [0, 1]_*$ be a countable subalgebra. Consider the expanded language $\mathcal{L}_\mathcal{C} = \mathcal{L} \cup \{\bar{r} : r \in C \setminus \{0, 1\}\}$ where we introduce a new constant for every element in $C \setminus \{0, 1\}$. We define $L_*(\mathcal{C})$ as the expansion of L_* in the language $\mathcal{L}_\mathcal{C}$ obtained by adding the so-called book-keeping axioms:*

$$\begin{aligned}\bar{r} \&\bar{s} &\leftrightarrow \bar{r} * \bar{s} \\ (\bar{r} \rightarrow \bar{s}) &\leftrightarrow \bar{r} \Rightarrow \bar{s}\end{aligned}$$

for every $r, s \in C$.

Notice that in this definition the book-keeping axioms $\bar{r} \wedge \bar{s} \leftrightarrow \overline{\min\{r, s\}}$ that would correspond to the other primitive connective in MTL, \wedge , are not present, since they are easily derivable in $L_*(\mathcal{C})$ as actually defined.

The algebraic counterparts of the $L_*(\mathcal{C})$ logics are defined in the natural way.

Definition 10.2. *Let $*$ be a left-continuous t-norm and let \mathcal{C} be a countable subalgebra of $[0, 1]_*$. An $L_*(\mathcal{C})$ -algebra is a structure*

$$\mathcal{A} = \langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \{\bar{r}^{\mathcal{A}} : r \in C\} \rangle$$

such that:

1. $\langle A, \&^{\mathcal{A}}, \rightarrow^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}} \rangle$ is an L_* -algebra, and
2. for every $r, s \in C$ the following identities hold:

$$\begin{aligned}\bar{r}^{\mathcal{A}} \&^{\mathcal{A}} \bar{s}^{\mathcal{A}} &= \overline{r * s}^{\mathcal{A}} \\ \bar{r}^{\mathcal{A}} \rightarrow^{\mathcal{A}} \bar{s}^{\mathcal{A}} &= \overline{r \Rightarrow s}^{\mathcal{A}}.\end{aligned}$$

The canonical standard $L_*(\mathcal{C})$ -chain is the algebra

$$[0, 1]_{L_*(\mathcal{C})} = \langle [0, 1], *, \Rightarrow, \min, \max, \{r : r \in C\} \rangle,$$

i. e. the $\mathcal{L}_\mathcal{C}$ -expansion of $[0, 1]_*$ where the truth-constants are interpreted by themselves.

Since the additional symbols added to the language are 0-ary, the condition of algebraizability given in Chapter 2 is trivially fulfilled. Therefore, $L_*(\mathcal{C})$ is also an algebraizable logic and its equivalent algebraic semantics is the variety of $L_*(\mathcal{C})$ -algebras, denoted as $\mathbb{L}_*(\mathcal{C})$. In particular this means that the logics $L_*(\mathcal{C})$ are strongly complete with respect to the variety of $L_*(\mathcal{C})$ -algebras. Furthermore, reasoning as in the MTL case, we can prove that all $L_*(\mathcal{C})$ -algebras are representable as a subdirect product of $L_*(\mathcal{C})$ -chains, hence we also have completeness of $L_*(\mathcal{C})$ with respect to $L_*(\mathcal{C})$ -chains.

Theorem 10.3. *For any $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_C}$, $\Gamma \vdash_{L_*(\mathcal{C})} \varphi$ if, and only if, $\{\psi \approx \bar{1} : \psi \in \Gamma\} \models_{\{L_*(\mathcal{C})\text{-chains}\}} \varphi \approx \bar{1}$.*

This general completeness with respect to chains, can be refined by using [35, Lemma 3.4.4], where Cintula proves a very general result for expansions of fuzzy logics with rational truth-constants. Adapted to our framework, it reads as follows.

Theorem 10.4 ([37]). *Let $*$ be a left-continuous t-norm such that L_* is strongly complete with respect to a class \mathbb{K} of L_* -chains. Then $L_*(\mathcal{C})$ is strongly complete with respect to the class of $L_*(\mathcal{C})$ -chains whose \mathcal{L} -reducts are in \mathbb{K} .*

Notice that when \mathbb{K} is the class of all L_* -chains, then this theorem does not provide anything new other than the result of Theorem 10.3. If \mathbb{K} is the class of standard L_* -chains, the condition that L_* should be strongly complete is very demanding. For instance if we restrict ourselves to continuous t-norm based logics, then only Gödel logic G satisfies this condition (SSC). If we consider logics of genuine left-continuous t-norms, then so far we can only additionally consider the NM logic and some WNM logics (see previous section).

Since all the logics $L_*(\mathcal{C})$ are expansions of MTL, sharing Modus Ponens as the only inference rule, they have the same local deduction-detachment theorem as MTL has. In fact, the proof for MTL or BL also applies here.

Theorem 10.5. *For every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}_C}$, $\Gamma, \varphi \vdash_{L_*(\mathcal{C})} \psi$ if, and only if, there is a natural $k \geq 1$ such that $\Gamma \vdash_{L_*(\mathcal{C})} \varphi^k \rightarrow \psi$.*

One can also show the following general result about the conservativity of $L_*(\mathcal{C})$ w.r.t. L_* .

Proposition 10.6. *$L_*(\mathcal{C})$ is a conservative expansion of L_* .*

Proof: Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ be arbitrary formulae and suppose that $\Gamma \vdash_{L_*(\mathcal{C})} \varphi$. Then, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{L_*(\mathcal{C})} \varphi$. By the above deduction theorem, there exists a natural k such that $\vdash_{L_*(\mathcal{C})} (\Gamma_0)^k \rightarrow \varphi$, identifying the set Γ_0 with the strong conjunction of all its formulae. By soundness, this implies that $\models_{[0,1]_{L_*(\mathcal{C})}} (\Gamma_0)^k \rightarrow \varphi$. Since the new truth-constants do not occur in $\Gamma_0 \cup \{\varphi\}$, we have $\models_{[0,1]_*} (\Gamma_0)^k \rightarrow \varphi$, and by SC of L_* , $\vdash_{L_*} (\Gamma_0)^k \rightarrow \varphi$, and hence $\Gamma \vdash_{L_*} \varphi$ as well. \square

In the rest of the chapter we will study the SC, FSSC and SSC properties for the logics with truth-constants $L_*(\mathcal{C})$, and also canonical standard completeness properties, i.e. SC, FSSC and SSC restricted to the canonical standard algebra.

10.2 Structure of $L_*(\mathcal{C})$ -chains

We have seen in Theorem 10.5 that the logics $L_*(\mathcal{C})$ are complete with respect to the $L_*(\mathcal{C})$ -chains. To study standard completeness results for $L_*(\mathcal{C})$ we need to obtain a deeper insight into $L_*(\mathcal{C})$ -chains. This is done in this section.

Next we assume $*$ is a left-continuous t-norm and \mathcal{C} is a countable subalgebra of $[0, 1]_*$.

Lemma 10.7. *For any $L_*(\mathcal{C})$ -chain $\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \{\bar{r}^{\mathcal{A}} : r \in C\} \rangle$, let $F_C(\mathcal{A}) = \{r \in C : \bar{r}^{\mathcal{A}} = \bar{1}^{\mathcal{A}}\}$ and $\overline{F_C(\mathcal{A})} = \{r \in C : \neg r \in F_C(\mathcal{A})\}$. Then:*

- (i) $F_C(\mathcal{A})$ is a filter of \mathcal{C} .
- (ii) The set $\{\bar{r}^{\mathcal{A}} : r \in C\}$ forms an L -subalgebra of \mathcal{A} , denoted as $\mathcal{C}^{\mathcal{A}}$, isomorphic to $\mathcal{C}/F_C(\mathcal{A})$, through the mapping $\bar{r}^{\mathcal{A}} \mapsto [r]_{\mathcal{A}}$, in such a way that

$$[1]_{\mathcal{A}} = F_C(\mathcal{A}) \text{ and } [0]_{\mathcal{A}} = \overline{F_C(\mathcal{A})},$$

where $[r]_{\mathcal{A}}$ denotes the equivalence class of $r \in C$ w.r.t. to the congruence defined by the filter $F_C(\mathcal{A})$.

Proof: (i) If $r \in F_C(\mathcal{A})$ and $s \in C$ with $s > r$, then $s \in F_C(\mathcal{A})$ because by the book-keeping axioms we have $\bar{s}^{\mathcal{A}} = \overline{\max(r, s)}^{\mathcal{A}} = \bar{r}^{\mathcal{A}} \vee \bar{s}^{\mathcal{A}} = \bar{1}^{\mathcal{A}}$. Moreover if $r, s \in F_C(\mathcal{A})$ then $r * s \in F_C(\mathcal{A})$ since $\overline{r * s}^{\mathcal{A}} = \bar{r}^{\mathcal{A}} \& \bar{s}^{\mathcal{A}} = \bar{1}^{\mathcal{A}}$. Therefore $F_C(\mathcal{A})$ is a filter.

(ii) Consider the function $f : \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{A}}$ defined by $f(r) = \bar{r}^{\mathcal{A}}$. It is clear that f is a surjective homomorphism and $\text{Ker } f = F_C(\mathcal{A})$, so $\mathcal{C}/F_C(\mathcal{A}) \cong \mathcal{C}^{\mathcal{A}}$. \square

In general, the equivalence classes of \mathcal{C} with respect to a filter F , i.e. the elements of \mathcal{C}/F , are difficult to describe, but some interesting cases can be indeed fully described. The next lemma refers to these cases, where we use the following notation:

CONT = $\{* \text{ is a continuous t-norm}\}$

CONT-fin = $\{* \in \mathbf{CONT} \mid * \text{ is a finite ordinal sum of the basic components}\}$

Lemma 10.8. *Let $* \in \mathbf{CONT} \cup \mathbf{WNM}$ and \mathcal{C} be a countable subalgebra of $[0, 1]_*$. For any $F \in \text{Fi}(\mathcal{C})$ we have:*

- (i) for any $r, s \notin F \cup \overline{F}$, $[r]_F = [s]_F$ iff $r = s$;
- (ii) moreover, if $* \in \mathbf{CONT}$ then $\overline{F} = \{0\}$.

Proof: If $* \in \mathbf{WNM}$, it follows from the description of quotients in WNM-chains given in Chapter 9. Suppose that $* \in \mathbf{CONT}$ and assume that $r < s \notin F \cup \overline{F}$ and $[r]_F = [s]_F$. Then $s \rightarrow r \in F$, but this is a contradiction since:

If $r, s \in (a_i, b_i)$ and $[a_i, b_i]$ is a Lukasiewicz component, then $s \rightarrow r$ is a nilpotent element belonging to F which implies that the minimum of the component belongs to F and therefore $[a_i, b_i] \subseteq F$, a contradiction.

If $r, s \in (a_i, b_i)$ and $[a_i, b_i]$ is a Product component, then $s \rightarrow r \in F$ which implies: If $r = 0$, then $0 \in F$, a contradiction; and if $r \neq 0$ then there exists n such that $r > (s \rightarrow r)^n$ and thus $r, s \in F$, a contradiction.

Finally if $r * s = \min\{r, s\}$ then $s \rightarrow r = r \in F$, a contradiction.

(ii) follows easily from the structure of the negation in standard BL-chains. \square

This lemma shows that the interpretation of the constants over a $L_*(\mathcal{C})$ -chain \mathcal{A} depends only on the filter $F_{\mathcal{C}}(\mathcal{A})$. Roughly speaking, if $i : C \rightarrow \{\bar{r}^{\mathcal{A}} : r \in C\}$ denotes that interpretation, i.e. $i(r) = \bar{r}^{\mathcal{A}}$ for all $r \in C$, then i maps truth-values r to $\bar{1}^{\mathcal{A}}$ or $\bar{0}^{\mathcal{A}}$ depending on whether $r \in F_{\mathcal{C}}(\mathcal{A})$ or $r \in \overline{F_{\mathcal{C}}(\mathcal{A})}$ respectively, and over the rest of the elements of C , i.e. those in $C \setminus (F_{\mathcal{C}}(\mathcal{A}) \cup \overline{F_{\mathcal{C}}(\mathcal{A})})$, i is a one-to-one mapping.

The standard chains of the variety $\mathbb{L}_*(\mathcal{C})$, i.e. the $L_*(\mathcal{C})$ -algebras over $[0, 1]$, are the key to obtain standard completeness results for the logic $L_*(\mathcal{C})$ when using the technique of partially embedding $L_*(\mathcal{C})$ -chains into standard ones. In order to know when such embeddings are possible, it is necessary to study the standard $L_*(\mathcal{C})$ -chains in more detail. This question is in fact related to describe the ways the truth-constants from C can be interpreted in $[0, 1]$ respecting the book-keeping axioms. We have seen in Lemmas 10.7 and 10.8 some necessary conditions showing the preeminent role of the set $Fi(\mathcal{C})$ of proper filters of \mathcal{C} plays in this question. Observe that each proper filter of \mathcal{C} is either of type $F^a = \{x \in C : x \geq a\}$ or of type $F^{>a} = \{x \in C : x > a\}$ for some $a \in C$.

One can wonder whether, given a filter $F \in Fi(\mathcal{C})$, there always exists a standard $L_*(\mathcal{C})$ -chain \mathcal{A} such that $F_{\mathcal{C}}(\mathcal{A}) = F$. Obviously, the simplest thing to look at is whether the algebra

$$[0, 1]_{L_*(\mathcal{C})}^F = \langle [0, 1], *, \Rightarrow, \min, \max, \{i_F(r) : r \in C\} \rangle,$$

where the mapping $i_F : C \rightarrow [0, 1]$ is defined as

$$i_F(r) = \begin{cases} 1, & \text{if } r \in F \\ 0, & \text{if } r \in \overline{F} \\ r, & \text{otherwise} \end{cases} \quad (10.1)$$

is always an $L_*(\mathcal{C})$ -algebra over $[0, 1]_*$, or in other words, whether the mapping i_F is always a proper interpretation of the truth-constants, in the sense of satisfying the book-keeping axioms.

It is easy to check that this is actually the case when $*$ \in **CONT**, and in such a case $[0, 1]_{L_*(\mathcal{C})}^F$ will be called *standard algebra of type F* . Moreover, when $*$ \in **CONT-fin**, one can show that these are all the standard chains over $[0, 1]_*$ one can define, in the sense that there are as many $L_*(\mathcal{C})$ -algebras over $[0, 1]_*$ (up to isomorphism) as proper filters of \mathcal{C} . To this end, first we need a technical lemma, which is related to the so-called Hion's Lemma (see e.g. [73, Lemma 4.1.6]).

Lemma 10.9. *Let C be a subset of $[0, 1]$ containing 0 and 1 and closed under the product of real numbers. Let $g : C \rightarrow [0, 1]$ satisfy $g(x \cdot y) = g(x) \cdot g(y)$ for all $x, y \in C$ and $g(x) < g(y)$ for all $x, y \in C$ such that $x < y$. Then there exists $\alpha \in \mathbb{R}^+$ such that $g(r) = r^\alpha$ for all $r \in C$.*

Proof: By the assumptions on g , we have for all $r, s \in C$, $r, s > 0$ and for all $i, j \in \mathbb{N}$:

- (i) if $r^i \leq s^j$ then $g(r)^i \leq g(s)^j$
- (ii) if $r^i \geq s^j$ then $g(r)^i \geq g(s)^j$

Using logarithms in statements (i) and (ii) we obtain the following equivalent statements for all $i, j \in \mathbb{N}$:

- (i') if $i \cdot \ln r - j \cdot \ln s \leq 0$ then $i \cdot \ln g(r) - j \cdot \ln g(s) \leq 0$
- (ii') if $i \cdot \ln r - j \cdot \ln s \geq 0$ then $i \cdot \ln g(r) - j \cdot \ln g(s) \geq 0$

or equivalently,

- (i'') if $\frac{i}{j} \geq \frac{\ln s}{\ln r}$ then $\frac{i}{j} \geq \frac{\ln g(s)}{\ln g(r)}$
- (ii'') if $\frac{i}{j} \leq \frac{\ln s}{\ln r}$ then $\frac{i}{j} \leq \frac{\ln g(s)}{\ln g(r)}$

The fact that these inequalities hold for all natural numbers i, j implies that

$$\frac{\ln s}{\ln r} = \frac{\ln g(s)}{\ln g(r)}$$

Indeed, if $\frac{\ln s}{\ln r} > \frac{\ln g(s)}{\ln g(r)}$, then there is a rational number $\frac{i}{j}$ such that $\frac{\ln s}{\ln r} > \frac{i}{j} > \frac{\ln g(s)}{\ln g(r)}$. This contradicts (ii'). Similarly, $\frac{\ln s}{\ln r} < \frac{\ln g(s)}{\ln g(r)}$ contradicts (i').

Finally, taking an arbitrary strictly positive $r \in C$ and letting $\alpha = \ln g(r)/\ln r$, the above equality leads to

$$g(s) = s^\alpha$$

for each $s \in C$. This ends the proof. \square

Proposition 10.10. *Let $* \in \mathbf{CONT}\text{-fin}$. Then:*

For any $F \in Fi(C)$, the algebra $[0, 1]_{L_(C)}^F$ is an $L_*(C)$ -algebra. Conversely, any standard $L_*(C)$ -chain whose \mathcal{L} -reduct is $[0, 1]_*$ is (up to isomorphism) an algebra $[0, 1]_{L_*(C)}^F$, for some $F \in Fi(C)$.*

More precisely: Let $X = \{[\mathcal{A}] : \mathcal{A} \text{ standard } L_(C)\text{-algebra over } [0, 1]_*\}$ be the set of isomorphism classes of $L_*(C)$ -algebras over $[0, 1]_*$. Then, the function $\Phi : X \rightarrow Fi(C)$ defined by $\Phi([\mathcal{A}]) = F_C(\mathcal{A})$ for every $[\mathcal{A}] \in X$ is a bijection.*

Proof: Given $F \in Fi(C)$, an easy computation shows that the algebra $[0, 1]_{L_*(C)}^F$ is an $L_*(C)$ -algebra. Φ is well-defined: if $\mathcal{A} \cong \mathcal{B}$, then it is clear that $\mathcal{C}^{\mathcal{A}} \cong \mathcal{C}^{\mathcal{B}}$, so $F_C(\mathcal{A}) = F_C(\mathcal{B})$.

Φ is clearly onto because $\Phi([0, 1]_{L_*(C)}^F) = F$. We must prove that Φ is also injective. Suppose that $\Phi(\mathcal{A}) = \Phi(\mathcal{B})$, i. e. $F_C(\mathcal{A}) = F_C(\mathcal{B})$. Then, we have $\mathcal{C}^{\mathcal{A}} \cong \mathcal{C}/F_C(\mathcal{A}) = \mathcal{C}/F_C(\mathcal{B}) \cong \mathcal{C}^{\mathcal{B}}$, and we want to conclude that $\mathcal{A} \cong \mathcal{B}$.

1. If $*$ is the Lukasiewicz t-norm, the only proper filter of C is $\{1\}$. Thus, $\mathcal{C}^{\mathcal{A}} \cong \mathcal{C}^{\mathcal{B}} (\cong C)$. Let h denote this isomorphism. If $h \neq Id$, then there is $a \neq h(a)$. Let $b = h(a)$. Taking the restriction of h , it is clear that the

generated subalgebras are also isomorphic, i. e. $\langle a \rangle \cong \langle b \rangle$, so a and b are either both rational or either both irrational (otherwise, the rational one would generate a finite subalgebra, and the irrational one would generate an infinite subalgebra). If a and b are irrational, then by [65, Proposition 2 and Theorem 3] $a = 1 - b$. Therefore one of them must be positive; suppose that it is a . Then $2a = 1$, so $2(1 - b) = 1$. But, due to the isomorphism, we also have $2b = 1$, a contradiction. If a and b are rational we reason analogously.

2. If $*$ is the minimum t-norm and $F_{\mathcal{C}}(\mathcal{A}) = F_{\mathcal{C}}(\mathcal{B})$ is any proper filter and $\mathcal{C}^{\mathcal{A}} \cong \mathcal{C}/F_{\mathcal{C}}(\mathcal{A}) = \mathcal{C}/F_{\mathcal{C}}(\mathcal{B}) \cong \mathcal{C}^{\mathcal{B}}$, then we can define a function h such that for every $r \in C \setminus F_{\mathcal{C}}(\mathcal{A})$, $h(\bar{r}^{\mathcal{A}}) = \bar{r}^{\mathcal{B}}$, and then we extend it to an isomorphism from \mathcal{A} to \mathcal{B} .
3. If $*$ is the product t-norm there are only two proper filters, $\{1\}$ and $C \setminus \{0\}$ and thus we have two types of $\Pi(\mathcal{C})$ -chains over $[0, 1]_{\Pi}$ corresponding to the cases that $F = \{1\}$ (the corresponding type of $\Pi(\mathcal{C})$ -chains are the ones such that for each pair $r < s$ in C , then $\bar{r}^{\mathcal{A}} < \bar{s}^{\mathcal{A}}$) and the case $C \setminus \{0\}$ (the corresponding type of $\Pi(\mathcal{C})$ -chains are such that $\bar{r}^{\mathcal{A}} = \bar{1}^{\mathcal{A}}$ for all $r \neq 0$). If $F_{\mathcal{C}}(\mathcal{A}) = F_{\mathcal{C}}(\mathcal{B}) = \{1\}$, then $\mathcal{C}^{\mathcal{A}} \cong \mathcal{C}^{\mathcal{B}} \cong \mathcal{C}$. By Lemma 10.9, there exist positive reals α and β such that $\bar{r}^{\mathcal{A}} = r^{\alpha}$ and $\bar{r}^{\mathcal{B}} = r^{\beta}$ for each $r \in C$. Therefore, the mapping $h : [0, 1] \rightarrow [0, 1]$ defined as $h(x) = x^{\beta/\alpha}$ defines an isomorphism from \mathcal{A} to \mathcal{B} . If $F_{\mathcal{C}}(\mathcal{A}) = F_{\mathcal{C}}(\mathcal{B}) = C \setminus \{0\}$, the result is trivial.
4. If $*$ is any continuous t-norm, then all possible proper filters are either of the form $[a, 1]$ where a is either in a Gödel component, or the minimum of a Łukasiewicz component or the minimum of a product component, or of the form $(a, 1]$ where a is either in a Gödel component or the minimum of a product component. The result is proved by applying the previous cases to each component of its decomposition not included in the filter.

□

For the case of $L_*(\mathcal{C})$ logics where $*$ \in **WNM-fin** the situation is not so simple. We illustrate the problem with an example. Let $*$ be the WNM t-norm represented at left hand side of Figure 9.4 and take $C = \mathbb{Q} \cap [0, 1]$. Let a be a positive involutive element such that $I_a^* \neq \{a\}$ and let F_a be the principal filter generated by a . Then the mapping $i_{F_a} : C \rightarrow [0, 1]$, defined as in expression (10.1), is not a proper interpretation of the truth-constants since for each $b \in I_a^*$, $\neg i(b) = \neg b = \neg a$ and $i(\neg b) = i(\neg a) = 0$, i.e. the book-keeping axioms are not satisfied and hence the algebra $[0, 1]_{L_*(\mathcal{C})}^F$ is not an $L_*(\mathcal{C})$ -algebra. Thus the mapping (10.1) used to interpret the truth-constants in the case of continuous t-norms does not always work in the case of a WNM t-norm.

In fact, for the case $*$ \in **WNM-fin**, if we want to associate to each filter $F \in Fi(\mathcal{C})$ a standard chain of $L_*(\mathcal{C})$ such that $F_{\mathcal{C}}(\mathcal{A}) = F$, we need to proceed in a different way. We will divide the job by cases.

1. If the classes of \mathcal{C}/F satisfy the condition that $\neg[r]_F = [0]_F$ implies $[r]_F = [1]_F$, then the interpretation used in the case of continuous t-norms works well and the chain $[0, 1]_{L_*(\mathcal{C})}^F$ is an $L_*(\mathcal{C})$ -chain like in the continuous case.
2. If in \mathcal{C}/F there are classes such that

$$[r]_F \neq [1]_F \text{ (that is, } r \notin F) \text{ and } \neg[r]_F = [0]_F,$$

then the mapping $i_F : C \rightarrow [0, 1]$ defined by expression (10.1) is not, in general, an interpretation as the example above proves.

Thus in this case, we consider two further subcases:

- (a) If $[0, 1]_*$ is such that $I_1^* \neq \{1\}$ (i.e. $\neg x = 0$ for some $x < 1$), then the mapping $i'_F : C \rightarrow [0, 1]$ defined by,

$$i'_F(r) = \begin{cases} 1, & \text{if } r \in F \\ 0, & \text{if } r \in \overline{F} \\ f(r), & \text{if } \neg r = 0 \text{ and } r \notin (F \cup \overline{F}) \\ r, & \text{otherwise} \end{cases} \quad (10.2)$$

where $f : \{r \in C \mid \neg r = 0, r \notin (F \cup \overline{F})\} \rightarrow I_1^*$ is an (arbitrary) one-to-one increasing mapping, is an interpretation which satisfies the book-keeping axioms. Then the algebra

$$[0, 1]_{L_*(\mathcal{C})}^F := \langle [0, 1], *, \Rightarrow_*, \min, \max, \{i'_F(r) : r \in C\} \rangle$$

is an $L_*(\mathcal{C})$ -chain over $[0, 1]_{**}$ chain of type F .

- (b) If $[0, 1]_*$ is such that $I_1^* = \{1\}$ (i.e. $\neg x = 0$ implies $x = 1$), then the mapping $i'_F : C \rightarrow [0, 1]$ defined in the previous case does not apply here since having $I_1^* = \{1\}$ makes impossible to define a one-to-one mapping f as required there. In this case we take as initial chain, not the standard chain $[0, 1]_*$, but the chain $([0, 1]_*)/F_a$ (which still belongs to the variety \mathbb{L}_*) where $a \in C$ is the greatest element in the constant intervals of $[0, 1]_*$. Notice that $[1]_{F_a} = [a, 1]$, $[0]_{F_a} = [0, \neg a]$ and $[r]_{F_a} = \{r\}$ for any $r \in (\neg a, a)$. Hence, $([0, 1]_*)/F_a$ is isomorphic to an L_* -chain $[-a, a]_{*'}'$ by identifying $[1]_{F_a}$ with a , $[0]_{F_a}$ with $\neg a$, and $[r]_{F_a}$ with r for all $r \in (\neg a, a)$, and by taking $*'$ as the obvious adaptation to the interval $[-a, a]$ of the original $*$. Now it is clear that $[-a, a]_{*'}'$ is such that $I_1^{*'} \neq \{1\}$ and therefore we can define a mapping $i''_F : C \rightarrow [-a, a]$ analogously to (10.2) which makes the algebra

$$\langle [-a, a], *', \Rightarrow_{*'}, \min, \max, \{i''_F(r) : r \in C\} \rangle$$

an $L_*(\mathcal{C})$ -chain. Finally, by means of an increasing linear transformation $h : [-a, a] \rightarrow [0, 1]$, it is easy to obtain an isomorphic $L_*(\mathcal{C})$ -chain over $[0, 1]$

$$[0, 1]_{L_*(\mathcal{C})}^F := \langle [0, 1], \circ, \Rightarrow_\circ, \min, \max, \{j_F(r) : r \in C\} \rangle$$

where $x \circ y = h(h^{-1}(x) * h^{-1}(y))$ and $j_F(r) = h(i_F''(r))$ for all $r \in C$. Notice that \circ needs not coincide with $*$.

Thus, we have the following corollary.

Corollary 10.11. *Let $*$ \in **CONT-fin** \cup **WNM-fin** and let \mathcal{C} be a countable subalgebra of $[0, 1]_*$. Then, for any filter $F \in \text{Fi}(\mathcal{C})$, there exists a standard $L_*(\mathcal{C})$ -chain \mathcal{A} such that $F_{\mathcal{C}}(\mathcal{A}) = F$, namely $\mathcal{A} = [0, 1]_{L_*(\mathcal{C})}^F$.*

Any standard $L_*(\mathcal{C})$ -chain \mathcal{A} such that $F_{\mathcal{C}}(\mathcal{A}) = F$ will be called from now on *standard $L_*(\mathcal{C})$ -chain of type F* .

10.3 Completeness results

In this section we will give completeness results for the logics $L_*(\mathcal{C})$ in the following particular cases:

1. When $*$ \in **CONT-fin** and \mathcal{C} is a countable subalgebra of $[0, 1]_*$ such that \mathcal{C} has elements in the interior of each component of the t-norm $*$, and in addition every $r \in C$ belonging to a Łukasiewicz component generates a finite MV-chain.
2. When $*$ \in **WNM-fin** and \mathcal{C} is a countable subalgebra of $[0, 1]_*$ such that has elements in the interior of each interval of the partition.

Thus, from now on we will assume that the algebra \mathcal{C} satisfies these conditions.

In the following subsection we will focus on strong and finite strong standard results while in the second subsection we will focus on the issue of canonical standard completeness.

10.3.1 About SSC and FSSC results

We start with a general result on strong standard completeness when $*$ \in **WNM-fin** which is consequence of Theorem 10.4 and the SSC results given in Corollary 9.33.

Theorem 10.12. *For every $*$ \in **WNM-fin** and every suitable \mathcal{C} , the logic $L_*(\mathcal{C})$ enjoys the SSC restricted to the family $\{[0, 1]_{L_*(\mathcal{C})}^F : F \in \text{Fi}(\mathcal{C})\}$.*

As particular cases of the above theorem we obtain that the logics $G(\mathcal{C})$ and $NM(\mathcal{C})$ enjoy the SSC restricted to the corresponding family $\{[0, 1]_{L_*(\mathcal{C})}^F : F \in \text{Fi}(\mathcal{C})\}$.

Notice that these results can never be improved to canonical SSC, as the following example shows.

Example 6. *For every non-trivial filter F (that exists in all these cases) and every $r \in F \setminus \{1\}$, the derivation*

$$(x \rightarrow y) \rightarrow \bar{r} \models y \rightarrow x$$

is valid in $[0, 1]_{L_(\mathcal{C})}$ but not in $[0, 1]_{L_*(\mathcal{C})}^F$.*

Observe that by Proposition 5.8 and Theorem 5.9 and being $L_*(C)$ a conservative expansion of L_* , the SSC is false for the logics $L_*(C)$ for each $*$ \in **CONT** when $*$ \neq min.

Since there is no result relating the FSSC for logics without truth-constants to the FSSC for the corresponding expanded logic with truth-constants, in order to study the FSSC we need to use the bridge result given in Theorem 5.4 i.e. we have to study partial embeddability for algebras with truth-constants.

Definition 10.13. *The logic $L_*(C)$ has the partial embeddability property if, and only if, for every filter $F \in Fi(C)$ and every subdirectly irreducible $L_*(C)$ -chain \mathcal{A} of type F , \mathcal{A} is partially embeddable into $[0, 1]_{L_*(C)}^F$.*

Obviously, the logics with truth-constants that enjoy the SSC restricted to the family of standard chains of type F , being F a proper filter of C , enjoy the partial embeddability property as well. Thus in the next theorem we consider cases that in general do not enjoy the SSC. For Łukasiewicz logic with rational truth-constants the problem has been already solved by Hájek.

Theorem 10.14 ([79]). *For every countable subalgebra $C \subseteq [0, 1]_L$ of rational numbers, the logic $L(C)$ enjoys the canonical FSSC.*

Open problem: In the previous theorem, is the condition $C \subseteq \mathbb{Q} \cap [0, 1]$ necessary?

The proof of the partial embeddability property for $\Pi(C)$ will take a lot of work. Let us denote by $[0, 1]_{\Pi(C)}^*$ the standard $\Pi(C)$ -chain corresponding to the filter $(0, 1]$. First, we will show that the variety generated by the $\Pi(C)$ -algebras on $[0, 1]$ is $\mathbf{V}([0, 1]_{\Pi(C)})$. Therefore, we need to show that $[0, 1]_{\Pi(C)}^*$ belongs to the variety generated by $[0, 1]_{\Pi(C)}$. In order to prove this in Theorem 10.17, we need a method to convert a nonsatisfying evaluation e of a $\Pi(C)$ -formula in $[0, 1]_{\Pi(C)}^*$ to a nonsatisfying evaluation e' of the same formula in $[0, 1]_{\Pi(C)}$. This is achieved in the following paragraphs concluded by the specific result in Proposition 10.16.

Let e be an evaluation of $\Pi(C)$ -formulae on the type Π algebra $[0, 1]_{\Pi(C)}^*$. In particular, for every $r \in C \setminus \{0\}$, we have $e(\bar{r}) = 1$. Consider the following set of evaluations e'_t on the canonical standard algebra $[0, 1]_{\Pi(C)}$, parametrized by positive real numbers $t \in \mathbb{R}^+$, defined as follows.

- $e'_t(\bar{r}) = r$ for every truth-constant symbol \bar{r} ,
- $e'_t(x) = (e(x))^t$ for every propositional variable x ,
- composite formulae are evaluated according to the operations in $[0, 1]_{\Pi(C)}$.

We are going to prove that if $e(\phi) < 1$, then $e'_t(\phi) < 1$ for every t large enough. We start by making the following remarks.

The set $[0, 1]^{\mathbb{R}^+}$ of all functions from \mathbb{R}^+ into $[0, 1]$ becomes a Π -algebra with the operations \cdot and \Rightarrow_{Π} defined pointwise and with the constant function 0 as bottom and the constant function 1 as top.

Let $F \subseteq [0, 1]^{\mathbb{R}^+}$ be the set of all functions $f : \mathbb{R}^+ \rightarrow [0, 1]$ satisfying the following condition:

(E) There are $0 < c \leq 1$ and $t_0 > 0$ such that $c \leq f(t)$ for all $t \geq t_0$.

It is immediate to verify that F is a filter of the Π -algebra $[0, 1]^{\mathbb{R}^+}$ (see e.g. [32, Lemma 1.5]). Hence the congruence relation defined by F on $[0, 1]^{\mathbb{R}^+}$, $f \sim g$ iff $f \Rightarrow_{\Pi} g \in F$ and $g \Rightarrow_{\Pi} f \in F$, turns out to be defined as

$f \sim g$ iff there are $0 < c, d \leq 1$ and $t_0 > 0$ such that $c \cdot f(t) \leq g(t) \leq f(t)/d$ for all $t > t_0$,

Indeed, if $c \leq f(t) \Rightarrow g(t)$ for $t > t_1$, then $c \cdot f(t) \leq g(t)$, and if $d \leq g(t) \Rightarrow f(t)$, then $d \cdot g(t) \leq f(t)$, for $t > t_2$. Therefore $c \cdot f(t) \leq g(t) \leq f(t)/d$, for $t > \max(t_1, t_2)$.

Lemma 10.15. *The congruence relation \sim satisfies:*

- (i) $f \sim 0$ iff there exists t_0 such that $f(t) = 0$ for all $t > t_0$
- (ii) If $f \sim g$ then $\lim_{t \rightarrow \infty} g(t) = 0$ iff $\lim_{t \rightarrow \infty} f(t) = 0$.

Proof: Both statements are straightforward using the above equivalence. \square

Proposition 10.16. *Let e and e'_t be defined as above. For every formula ϕ let $g_\phi(t) = (e(\phi))^t$ and $f_\phi(t) = e'_t(\phi)$. Then we have $f_\phi \sim g_\phi$. In particular, if $e(\phi) < 1$, then $\lim_{t \rightarrow \infty} e'_t(\phi) = 0$.*

Proof: Let us proceed by induction on the complexity of ϕ .

1. Constants.

$r = 0$. $g_{\bar{0}}(t) = e(\bar{0})^t = 0$ and $f_{\bar{0}}(t) = e'_t(0) = 0$, and $0 \sim 0$.

$r > 0$. $g_{\bar{r}}(t) = (e(\bar{r}))^t = 1^t = 1$ and $f_{\bar{r}}(t) = e'_t(\bar{r}) = r$, and obviously $r \sim 1$.

2. Variables.

Direct consequence of the definition ($f_x(t) = g_x(t)$).

3. $\phi = (\psi_1 \& \psi_2)$.

$g_{\psi_1 \& \psi_2}(t) = e(\psi_1 \& \psi_2)^t = e(\psi_1)^t \cdot e(\psi_2)^t = g_{\psi_1}(t) \cdot g_{\psi_2}(t)$.

$f_{\psi_1 \& \psi_2}(t) = e'_t(\psi_1 \& \psi_2) = e'_t(\psi_1) \cdot e'_t(\psi_2) = f_{\psi_1}(t) \cdot f_{\psi_2}(t)$.

Since \sim is a congruence, if we suppose that $f_{\psi_1} \sim g_{\psi_1}$ and $f_{\psi_2} \sim g_{\psi_2}$, we can conclude that $f_{\psi_1 \& \psi_2} \sim g_{\psi_1 \& \psi_2}$.

4. $\phi = (\psi_1 \rightarrow \psi_2)$.

$g_{\psi_1 \rightarrow \psi_2}(t) = e(\psi_1 \rightarrow \psi_2)^t = (e(\psi_1) \Rightarrow e(\psi_2))^t = e(\psi_1)^t \Rightarrow e(\psi_2)^t = g_{\psi_1}(t) \Rightarrow g_{\psi_2}(t)$.

$f_{\psi_1 \rightarrow \psi_2}(t) = e'_t(\psi_1 \rightarrow \psi_2) = e'_t(\psi_1) \Rightarrow e'_t(\psi_2) = f_{\psi_1}(t) \Rightarrow f_{\psi_2}(t)$.

Using again the fact that \sim is a congruence, from the hypothesis $f_{\psi_1} \sim g_{\psi_1}$ and $f_{\psi_2} \sim g_{\psi_2}$, we obtain $f_{\psi_1 \rightarrow \psi_2} \sim g_{\psi_1 \rightarrow \psi_2}$.

The first statement of the proposition is proved. The second statement follows from the first statement and (ii) of Lemma 10.15. \square

Theorem 10.17. $[0, 1]_{\Pi(\mathcal{C})}^* \in \mathbf{V}([0, 1]_{\Pi(\mathcal{C})})$, hence the variety generated by the class of $\Pi(\mathcal{C})$ -algebras over the unit real interval $[0, 1]$ is $\mathbf{V}([0, 1]_{\Pi(\mathcal{C})})$.

Proof: Let φ be not valid in $[0, 1]_{\Pi(\mathcal{C})}^*$. There exists an evaluation e on $[0, 1]_{\Pi(\mathcal{C})}^*$ such that $e(\varphi) < 1$. By the above proposition, $\lim_{t \rightarrow \infty} e'_t(\varphi) = 0$ as well, hence for every large enough t , $e'_t(\varphi) < 1$. Since e'_t is an evaluation on $[0, 1]_{\Pi(\mathcal{C})}$, φ is not valid in $[0, 1]_{\Pi(\mathcal{C})}$. \square

To show that the logic $\Pi(\mathcal{C})$ has the partial embedding property we introduce some notation. Given a linearly ordered $\Pi(\mathcal{C})$ -algebra \mathcal{A} and a finite subset E of A , denote by C_E the set $\{r \in C \mid \bar{r}^A \in E\}$. Let \widetilde{C}_E be the Π -algebra generated by C_E . Note that the Π -algebra generated by E is naturally a $\Pi(\widetilde{C}_E)$ -algebra. Let \widetilde{C}_E^* be \widetilde{C}_E without 0.

In the proof of the next proposition we make use of the ordered Abelian group $(\mathcal{R}^+)_{lex}^k$ obtained as the lexicographic product of k copies of the multiplicative groups of positive reals.

Proposition 10.18. Let \mathcal{A} be a linearly ordered $\Pi(\mathcal{C})$ -algebra of type I and let E be a finite subset of \mathcal{A} . Let \mathcal{A}_E be the linearly ordered $\Pi(\widetilde{C}_E)$ -algebra generated by E . Then \mathcal{A}_E is isomorphic to a $\Pi(\widetilde{C}_E)$ -algebra \mathcal{D} such that the following is satisfied:

- $\mathcal{D} = \mathbf{P}(\mathcal{G})$ with \mathcal{G} being a subgroup of $(\mathcal{R}^+)_{lex}^k$, where k is an integer.
- there is an integer l and a real number $\alpha > 0$ such that for every $r \in \widetilde{C}_E^*$, we have $\bar{r}^{\mathcal{D}} = \omega_{k,l}(r^\alpha)$,

where, for any $x \in (0, 1]$ and natural $1 \leq l \leq k$, $\omega_{k,l}(x) = (1, \dots, 1, x, 1, \dots, 1) \in (\mathbb{R}^+)^k$ with x being at coordinate with index l .

Proof: Taking \mathcal{A}_E as a Π -algebra, there is a linearly ordered Abelian group \mathcal{G}' such that $\mathcal{A}_E = \mathbf{P}(\mathcal{G}')$, i.e. $\mathcal{A}_E \setminus \{0\}$ is the negative cone of a linearly ordered group \mathcal{G}' . Since \mathcal{A}_E is finitely generated, so is \mathcal{G}' . Hence, applying Theorem 2.23 there is a natural k such that \mathcal{G}' is isomorphic to a subgroup \mathcal{G} of $(\mathcal{R}^+)_{lex}^k$ (see Chapter 2 for the definition of $(\mathcal{R}^+)_{lex}^k$ and the result).

Then \mathcal{A}_E is also isomorphic (through a mapping ι) to $\mathbf{P}(\mathcal{G})$ as Π -algebras. For every $r \in \widetilde{C}_E$ define $\bar{r}^{\mathbf{P}(\mathcal{G})} = \iota(\bar{r}^{\mathcal{A}_E})$. Using this, $\mathbf{P}(\mathcal{G})$ is a $\Pi(\widetilde{C}_E)$ -algebra isomorphic to \mathcal{A}_E . Therefore, for simplicity, we may assume from now that $\mathcal{A}_E = \mathbf{P}(\mathcal{G})$.

Since \widetilde{C}_E is an Archimedean Π -algebra, there is a unique $l \leq k$ such that for each element \bar{r}^A , $r < 1$ and $r \in \widetilde{C}_E^*$, we have $\bar{r}^A = (1, \dots, 1, a_l, \dots, a_k)$ with $a_l < 1$. Indeed, suppose $r, s \in \widetilde{C}_E^*$ such that $s < r < 1$ and $\bar{r}^A = (1, \dots, 1, a_i, \dots, a_k)$ and $\bar{s}^A = (1, \dots, 1, b_j, \dots, b_k)$ with $i > j$. There is a natural m such that $r^m < s$,

but obviously $(\bar{r}^A)^m = (1, \dots, 1, (a_i)^m, \dots, (a_k)^m) >_{lex} (1, \dots, 1, b_j, \dots, b_k) = \bar{s}^A$, contradiction.

Let $f_1, \dots, f_k : \widetilde{C_E}^* \rightarrow \mathbb{R}^+$ be functions such that for each $r \in \widetilde{C_E}^*$ we have $\bar{r}^A = (f_1(r), \dots, f_k(r))$. Due to the validity of the book-keeping axioms, for each i , $f_i : \widetilde{C_E}^* \rightarrow \mathbb{R}^+$ is a homomorphism for the product. According to the above paragraph, f_l is the first of the functions f_i which is not the constant 1. Since the algebra \mathcal{A}_E is a $\Pi(\widetilde{C_E})$ -algebra of type I and, by the previous paragraph, $f_l(r) < 1$ for every $r < 1$, $r \in \widetilde{C_E}^*$, f_l is one-to-one and preserves the order (indeed if $f_l(r) = f_l(s)$ for some $r \geq s$, then $f_l(r \Rightarrow s) = 1$). Hence, by Lemma 10.9, f_l is a power and we have

$$\bar{r}^A = (1, 1, \dots, r^\alpha, f_{l+1}(r), \dots, f_k(r)) \quad (*)$$

for some real $\alpha > 0$.

Now let $M = \{x_l \mid (x_1, \dots, x_l, x_{l+1}, \dots, x_k) \in \mathcal{G}\}$ be the set of all l -components of elements of \mathcal{G} . By its definition, M with the multiplication is a subgroup of \mathbb{R}^+ which is generated by the set of r^α 's, for $r \in \widetilde{C_E}^*$, and additionally by a finite number of values x_l coming from the elements of E . Now, define mappings $g_{l+1}, \dots, g_k : M \rightarrow \mathbb{R}^+$ as follows:

1. put $g_j(r^\alpha) = f_j(r)$ for all $r \in \widetilde{C_E}^*$
2. using Lemma 2.24, extend g_j to the subgroup generated by $\widetilde{C_E}^*$
3. finally, applying Lemma 2.25 for the l -component of each element of E , which is not an interpretation of an element of C_E , extend g_j to the whole M .

As a result, we obtain a homomorphism g_j from M to \mathbb{R}^+ for each $j \in \{l+1, \dots, k\}$.

Finally, define a new mapping $h : \mathcal{G} \rightarrow (\mathbb{R}^+)^k$ by putting

$$h((x_1, \dots, x_l, x_{l+1}, \dots, x_k)) = (x_1, \dots, x_l, x_{l+1}/g_{l+1}(x_l), \dots, x_k/g_k(x_l)).$$

We claim that, so defined, h is a monomorphism. Indeed, since the g_j 's are homomorphisms for the product on M , h is a homomorphism for the product on \mathcal{G} as well. If two elements of \mathcal{G} differ in x_i for $i \leq l$, then their images are ordered in the same way, since the first l coordinates are not changed by h . If two elements of \mathcal{G} agree in the first l coordinates and the first different coordinate is x_i for $i > l$, then their images are ordered in the same way, since x_i is again the first differing coordinate and x_i is divided by the same number in both images.

Therefore, $h(\mathcal{G})$ is a subgroup of $(\mathbb{R}^+)^k_{lex}$ which is isomorphic to \mathcal{G} . Consider the Π -algebra $\mathcal{D} = \mathbf{P}(h(\mathcal{G}))$. By construction of h , we have $h(\bar{r}^A) = \omega_{k,l}(r^\alpha)$ for every $r \in \widetilde{C_E}^*$. Hence, by defining $\bar{r}^{\mathcal{D}} = h(\bar{r}^A) = \omega_{k,l}(r^\alpha)$ for every $r \in \widetilde{C_E}^*$, \mathcal{D} becomes a $\Pi(\widetilde{C_E})$ -algebra, and moreover, \mathcal{D} is isomorphic to \mathcal{A}_E . This ends the proof of Proposition 10.18. \square

In the following we show that there is a partial isomorphism of any $\Pi(\widetilde{\mathcal{C}}_E)$ -algebra of the special form guaranteed by Proposition 10.18 into the canonical standard $\Pi(\mathcal{C})$ -algebra.

Proposition 10.19. *Let \mathcal{G} be subgroup of $(\mathcal{R}^+)_{lex}^k$ such that $\mathcal{D} = \mathbf{P}(\mathcal{G})$ is a $\Pi(\mathcal{C})$ -algebra, with $\bar{r}^{\mathcal{D}} = \omega_{k,l}(r^\alpha)$ for every $r \in C^*$, for some natural l and positive real α , and $\bar{0}^{\mathcal{D}} = (0, \cdot, 0)$. Then for every finite subset E of \mathcal{D} there is a mapping $q : E \rightarrow [0, 1]$ satisfying the following four conditions*

- (i) q preserves the order,
- (ii) $q(\bar{r}^{\mathcal{D}}) = r$ for all $r \in C_E$,
- (iii) If $x, y, x * y \in E$ then $q(x) \cdot q(y) = q(x * y)$.
- (iv) If $x, y, x \Rightarrow y \in E$ then $q(x) \Rightarrow_{\Pi} q(y) = q(x \Rightarrow y)$.

Proof: The candidates for q are restrictions to E of functions $g : \mathcal{G} \rightarrow \mathbb{R}^+$ of the form

$$g((x_1, x_2, \dots, x_k)) = (x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \dots \cdot x_k^{\varepsilon_k})^\beta,$$

where $\varepsilon_i, \beta > 0$. Each of these functions is a homomorphism w.r.t. the product of \mathcal{G} . Hence, for every choice of ε_i and β , the restriction of g to E satisfies (iii). By the assumption, for every $r \in C^*$ $\bar{r}^{\mathcal{D}} = \omega_{k,l}(r^\alpha)$. Hence, for every choice of ε_i and β , we have $g(\bar{r}^{\mathcal{D}}) = r^{\alpha \cdot \varepsilon_l \cdot \beta}$, where $\alpha \cdot \varepsilon_l \cdot \beta > 0$. By choosing $\beta = 1/(\alpha \cdot \varepsilon_l)$, we obtain that the restriction of g to E satisfies (ii).

Let us prove that it is possible to choose the ε_i in such a way that the restriction of g to E satisfies (i). Let us classify the pairs of distinct values in E according to the first index i_0 , where the values differ. Pairs which satisfy $i_0 = k$ are ordered correctly for any positive value of ε_k . Pairs satisfying $i_0 = k - 1$ may be put into the right order by choosing $\varepsilon_{k-1} = 1$ and ε_k small enough to guarantee that the difference (measured as a ratio) in the $(k - 1)$ -th coordinate is always larger than the difference in the k -th coordinate. In fact, if the exponents $\varepsilon_{k-1} = 1, \varepsilon_k$ guarantee the right order of the pairs with $i_0 = k - 1$, then the exponents $\varepsilon_{k-1} = t, t \cdot \varepsilon_k$, for any positive t , guarantee the order as well. Hence, when it is necessary to put the pairs with $i_0 = k - 2$ into the right order, we choose $\varepsilon_{k-2} = 1$ and t small enough so that the difference in the $(k - 2)$ -th coordinate is always larger than the differences contributed by $(k - 1)$ -th and k -th coordinates. Since we preserve the ratio between ε_{k-1} and ε_k , we do not destroy the already correct order of pairs with $i_0 = k - 1$. We proceed in a similar way for pairs with smaller and smaller i_0 .

The condition (iv), the preservation of existing implications in E , is a consequence of h being order preserving (i) and the preservation of existing products (iii). \square

Theorem 10.20. *Let \mathcal{A} be a linearly ordered $\Pi(\mathcal{C})$ -algebra and let E be a finite subset of \mathcal{A} . Then there exists a one-to-one mapping $h : E \rightarrow [0, 1]$ satisfying the following conditions:*

- (i) h preserves the order,
- (ii) $h(\bar{r}^{\mathcal{A}}) = r$ for all $r \in C_E$,
- (iii) If $x, y, z \in E$ and $z = x * y$ then $h(x) \cdot h(y) = h(z)$.
- (iv) If $x, y, z \in E$ and $z = x \Rightarrow y$ then $h(x) \Rightarrow_{\Pi} h(y) = h(z)$.

	$G(\mathcal{C})$	$\Pi(\mathcal{C})$	$L(\mathcal{C})$	$L_*(\mathcal{C})$, for other $*$ \in CONT-fin
SC	Yes	Yes	Yes	Yes
FSSC	Yes	Yes	Yes	Yes
SSC	Yes	No	No	No
Canonical FSSC	No	No	Yes	No
Canonical SSC	No	No	No	No

Table 10.1: (Finite) strong standard completeness results for logics of a t-norm from **CONT-fin**.

Proof: Let \mathcal{D} be the algebra guaranteed by Proposition 10.18 applied to \mathcal{A}_E . Let E' be the image of E under the isomorphism between \mathcal{A}_E and \mathcal{D} . Applying Proposition 10.19 to \mathcal{D} and E' with $C = \bar{C}_E$, we obtain an embedding q , whose composition with the above isomorphism has the required properties of h . \square

Now we can finally prove the partial embeddability property for every $L_*(\mathcal{C})$, when $*$ \in **CONT-fin** and \mathcal{C} satisfies the conditions required at the beginning of the section.

Theorem 10.21. *For every $*$ \in **CONT-fin** and every suitable \mathcal{C} , the logic $L_*(\mathcal{C})$ enjoys the partial embeddability property, and therefore it has the FSSC restricted to the family $\{[0, 1]_{L_*(\mathcal{C})}^F : F \in Fi(\mathcal{C})\}$.*

Proof: Suppose that $[0, 1]_* = \bigoplus_{i=1}^n \mathcal{A}_i$. By Theorem 4.52 we know that the subdirectly irreducible chains of $\mathbf{V}([0, 1]_*)$ are members of $\mathbf{HSP}_U(\mathcal{A}_1) \cup (\mathbf{ISP}_U(\mathcal{A}_1) \oplus \mathbf{HSP}_U(\mathcal{A}_2)) \cup \dots \cup (\bigoplus_{i=1}^{n-1} \mathbf{ISP}_U(\mathcal{A}_i) \oplus \mathbf{HSP}_U(\mathcal{A}_n))$. Knowing this structure of \mathcal{A} as ordinal sum of the three basic components and taking into account that for every Łukasiewicz component \mathcal{A}_i , the condition that every $r \in C \cap \mathcal{A}_i$ generates a finite MV-chain amounts to say that in the isomorphic copy of this component as $[0, 1]_L$, every $r \in C \cap \mathcal{A}_i$ is isomorphically mapped to a rational number, the partial embeddability property for the three basic components gives the result. \square

Observe that in the Łukasiewicz case the subalgebra of constants \mathcal{C} has a unique proper filter $F = \{1\}$ and thus the logic enjoys the canonical FSSC. Moreover, the Example 6 also shows that the rest of the logics do not enjoy the canonical FSSC.

All these results are collected in Tables 10.1 and 10.2.

10.3.2 About canonical standard completeness

From the results of the last sections, we already know that all the considered logics enjoy the SC restricted to the family of standard chains associated to proper filters of \mathcal{C} , i.e, their theorems are exactly the common tautologies of the chains of the family $\{[0, 1]_{L_*(\mathcal{C})}^F : F \in Fi(\mathcal{C})\}$. But although the logics

	$G(\mathcal{C})$	$NM(\mathcal{C})$	$L_*(\mathcal{C})$, for other $*$ \in WNM-fin
SC	Yes	Yes	Yes
FSSC	Yes	Yes	Yes
SSC	Yes	Yes	Yes
Canonical FSSC	No	No	No
Canonical SSC	No	No	No

Table 10.2: (Finite) strong standard completeness results for logics of a t-norm from **WNM-fin**.

considered in the last sections have not in general the canonical SSC or the canonical FSSC (only $L(\mathcal{C})$ when $\mathcal{C} \subseteq \mathbb{Q} \cap [0, 1]$ enjoys it), some of them still enjoy the canonical SC, i. e. their theorems are exactly the tautologies of their corresponding canonical standard algebra. In order to prove it, we need to show that tautologies of the canonical standard chain are a subset of the tautologies of each one of the standard chains associated to each proper filter of \mathcal{C} . Of course, this is the case of $L(\mathcal{C})$ since it has the canonical FSSC. We will study this problem by cases in next subsections.

The case of **WNM-fin** t-norms

The question for the canonical SC for **WNM-fin** t-norms is fully answered. Some cases are proved to be canonical standard complete, while in the other cases we provide a counterexample showing that they are not canonical standard complete. In fact in [55] it is proved that the expansions of Gödel logic, NM logic and the logics corresponding to the t-norms \otimes_c and \star_c depicted in Figure 9.3 enjoy the canonical SC². Here we give a new unified and simpler proof.

Theorem 10.22. *If $*$ \in **WNM-fin** is such that its negation on the set of positive elements is either both involutive and continuous, or it is identically 0, then $L_*(\mathcal{C})$ enjoys the canonical SC*

Proof: Suppose φ is a tautology with respect to $[0, 1]_{L_*(\mathcal{C})}$. We will prove that φ is also a tautology with respect to $[0, 1]_{L_*(\mathcal{C})}^F$ for each $F \in Fi(\mathcal{C})$, which implies that $\vdash_{L_*(\mathcal{C})} \varphi$. Let e be an interpretation over the chain $[0, 1]_{L_*(\mathcal{C})}^F$. Suppose that \mathcal{A} is the finite algebra generated by $\{e(\psi) \mid \psi \text{ subformula of } \varphi\}$ and $\alpha = \min\{r \in F \mid \bar{r} \text{ occurs in } \varphi\}$. Suppose that $f : (\neg\alpha, \alpha) \rightarrow (0, 1)$ is a bijection such that $f(r) = r$ for all $r \notin F \cup \bar{F}$ such that \bar{r} occurs in φ and f is a homomorphism from \mathcal{A} to the canonical standard chain. Then define an evaluation e' on the canonical standard chain defined by $e'(p) = f^{-1}(e(p))$ if p is a propositional variable that appears in φ and $e'(p) = 1$ otherwise. Since φ is a tautology for the canonical standard chain, $e'(\varphi) = 1$. Take the algebra $[0, 1]_*/F_\alpha$ where F_α is the principal filter generated by α . By hypothesis this algebra is isomorphic

²In [55] it is wrongly claimed that the expansions $L_*(\mathcal{C})$ for $*$ $= \odot_c$ (see Figure 9.3) were also canonical standard complete, in Example 7 we provide a counter-example.

to $[0, 1]_*$. Define the evaluation e'' on the quotient algebra obtained from e' and it obviously satisfies $e''(\varphi) = [1]_{F_\alpha}$. But a simple computation shows that the algebra \mathcal{B} generated by $\{e''(\psi) \mid \psi \text{ subformula of } \varphi\}$ is isomorphic to \mathcal{A} and $e''(\varphi)$ over the quotient algebra corresponds to $e(\varphi)$ over the chain $[0, 1]_{L_*(C)}^F$ and thus $e(\varphi) = 1$. \square

Actually, the only expansions of logics L_* with $*$ \in **WNM-fin** that enjoy the canonical SC are those falling under the hypotheses of last theorem. This is proved below by showing counterexamples for the remaining cases, where $p(x)$ and $n(x)$ denote the terms $x \vee \neg x$ and $x \wedge \neg x$ respectively, as defined in Chapter 4.

Example 7. *Let $*$ \in **WNM-fin** not falling under the hypotheses of last theorem. We distinguish the following three cases:*

- *Suppose the negation is continuous on the set of positive elements and the only constant interval formed by positive elements is I_1 . In such a case, there is an interval I of involutive positive elements, followed by I_1 . Take a truth-constant b in the interior of I . Then the formula,*

$$(\neg \neg p(x) \rightarrow p(x)) \vee (\bar{b} \rightarrow p(x))$$

is a tautology for $[0, 1]_{L_(C)}$ and it is not a tautology for $[0, 1]_{L_*(C)}^F$ for any F containing b . Take into account that in $[0, 1]_{L_*(C)}$ a positive element is either involutive or greater than b .*

- *Suppose the negation is continuous on the set of positive elements and there is some constant interval formed by positive elements different from I_1 (this is the case of the family of t-norms \odot_c in Figure 9.3). Let b be the minimum involutive positive element with a non-trivial associated interval. Then the formula,*

$$(\neg \neg p(x) \rightarrow p(x)) \vee (\neg p(x) \rightarrow \bar{b})$$

is a tautology for $[0, 1]_{L_(C)}$ and it is not a tautology for $[0, 1]_{L_*(C)}^F$ for any F containing b . Notice that in this case $[0, 1]_{L_*(C)}^F$ is such that either a positive element is involutive or its negation is not greater than $\neg b$.*

- *Suppose the negation is continuous on the set of positive elements. Let b be the minimum discontinuity point of the negation function in the set of positive elements. Then $I_{\neg b}$ is the greatest constant interval in the negative part with biggest element $\neg b$ and not containing the fixpoint. Then take*

$$(\neg \neg n(x) \rightarrow n(x)) \vee (\neg n(x) \rightarrow \neg \neg n(x)) \vee (n(x) \rightarrow \bar{b})$$

is a tautology for $[0, 1]_{L_(C)}$ and it is not a tautology for $[0, 1]_{L_*(C)}^F$ for any F containing b . Notice that in $[0, 1]_{L_*(C)}$ a negative element is either involutive or belongs to a constant interval whose greatest element is the fixpoint (if it exists) or it is less or equal than $\neg b$.*

These three examples prove that a rather large family of expansions of the logic of a t-norm from **WNM-fin** with truth-constants do not enjoy the canonical SC. In fact, only the following cases are not included in the previous examples:

- when the set of positive elements defines an involutive interval of the partition $(\text{NM}, \star_c$ of Figure 9.3).
- when the set of positive elements defines a constant interval of the partition $(\text{G}, \otimes_c$ of Figure 9.3).

The case of continuous t-norms

For the case of expansions Łukasiewicz logic with truth-contrants, Hájek's result in [79] can be put as follows.

Theorem 10.23 ([79]). $\text{L}(\mathcal{C})$ has the canonical SC.

For the expansions of Gödel logic it has been already proved in Theorem 10.22.

Theorem 10.24. $\Pi(\mathcal{C})$ has the canonical SC.

Proof: Let φ be a $\Pi(\mathcal{C})$ formula such that $\not\models_{\Pi(\mathcal{C})} \varphi$. We can further assume φ contains some truth constant \bar{r} with $0 < r < 1$ as subformula, otherwise the standard completeness of Product Logic does the job. By general completeness, there is a linearly ordered $\Pi(\mathcal{C})$ -algebra \mathcal{A} and an evaluation e on \mathcal{A} such that $e(\varphi) < \bar{1}^{\mathcal{A}}$. The task is to find an evaluation e' on the canonical standard $\Pi(\mathcal{C})$ -algebra $[0, 1]_{\Pi(\mathcal{C})}$ such that $e'(\varphi) < 1$. Let $E = \{e(\psi) \mid \psi \text{ is a subformula of } \varphi\} \cup \{\bar{0}^{\mathcal{A}}, \bar{1}^{\mathcal{A}}\}$. We consider the following cases:

Case 1: $F_{\mathcal{C}}(\mathcal{A}) = \{1\}$.

By applying Theorem 10.20 we obtain a partial embedding h of E into $[0, 1]$. Now define a $[0, 1]_{\Pi(\mathcal{C})}$ -evaluation e' by putting

$$e'(x) = \begin{cases} h(e(p)), & \text{if } x \text{ is a prop. variable in } \varphi \\ \text{arbitrary}, & \text{otherwise} \end{cases}$$

It is easy to check then, by the properties of h , that $e'(\varphi) = h(e(\varphi)) < 1$.

Case 2: $F_{\mathcal{C}}(\mathcal{A}) = (0, 1]$.

By the well-known results of Π -algebras (see [31]), there is a partial embedding f of E into the standard Π -algebra $[0, 1]_{\Pi}$ and the evaluation e' on $[0, 1]_{\Pi}$ defined as follows

$$e'(p) = \begin{cases} f(e(p)), & \text{if } p \text{ is a propositional variable in } \varphi \\ \text{arbitrary}, & \text{otherwise} \end{cases}$$

is such that $e'(\varphi^*) < 1$, where φ^* is the Π -formula obtained from φ by replacing all truth-constants \bar{r} with $0 < r < \bar{1}$. Now, the evaluation e'' on

$[0, 1]_{\Pi(C)}^*$, such that $e''(p) = e'(p)$ for all propositional variables p satisfies $e'(\varphi^*) = e''(\varphi) < 1$. Then, by Theorem 10.17, there is also an evaluation e''' on the canonical standard $\Pi(C)$ -algebra $[0, 1]_{\Pi(C)}$ such that $e'''(\varphi) < 1$. This ends the proof of Case 2 and hence of the theorem as well.

□

But the canonical SC is not valid in general for any logic $L_*(\mathcal{C})$ with $*$ **CONT-fin**. It will be shown by providing counterexamples, i. e. by exhibiting in each case a suitable formula φ that is a tautology of the canonical standard algebra $[0, 1]_{L_*(\mathcal{C})}$ but not of the algebra $[0, 1]_{L_*(\mathcal{C})}^F$ for some proper filter F of \mathcal{C} . Suppose that the first component of $[0, 1]_*$ is defined on the interval $[0, a]$.

1. If $[0, 1]_* = [0, a]_L \oplus \mathcal{A}$ and $a \in C$, then an easy computation shows that the formula

$$\bar{a} \rightarrow (\neg\neg x \rightarrow x)$$

is valid in the canonical standard algebra but it is not valid in the standard chain $[0, 1]_{L_*(\mathcal{C})}^F$ defined by the filter $F = [a, 1] \cap C$ (where \bar{a} is interpreted as 1).

2. If $[0, 1]_* = [0, a]_{\Pi} \oplus \mathcal{A}$, take $b \in C \cap (0, a)$. Then an easy computation shows that the formula

$$\bar{b} \rightarrow \neg x \vee ((x \rightarrow x \& x) \rightarrow x)$$

is valid in the canonical standard algebra but it is not valid in the standard chain $[0, 1]_{L_*(\mathcal{C})}^F$ defined by the filter $F = (0, 1] \cap C$ (where \bar{b} is interpreted as 1).

3. If $[0, 1]_* = [0, a]_G \oplus \mathcal{A}$, take b as any element of $C \cap (0, a)$. Then the formula

$$\bar{b} \rightarrow (x \rightarrow x \& x)$$

is valid in the canonical standard algebra but it is not valid in the standard chain $[0, 1]_{L_*(\mathcal{C})}^F$ defined by the filter $F = [b, 1] \cap C$ (where \bar{b} is interpreted as 1).

Observe that for a t-norm whose decomposition begins with two copies of Łukasiewicz t-norm, the idempotent element a separating them has to belong to the truth-constants subalgebra \mathcal{C} . Indeed, take into account that, by assumption, C must contain a non idempotent element c of the second component and for this element there exists a natural number n such that $c^n = a$ and thus $a \in C$. Hence this case is subsumed in the above first item.

The remaining cases (when the first component is Łukasiewicz but its upper bound a does not belong to C) are studied by cases:

- (1) If $[0, 1]_* = [0, a]_L \oplus [a, 1]_G$ or $[0, 1]_* = [0, a]_L \oplus [a, 1]_\Pi$, then the logic $L_*(\mathcal{C})$ has the canonical SC. Actually, in that case the filters of \mathcal{C} are the same as the filters of $C \cap [a, 1]_G$ or $C \cap [a, 1]_\Pi$ respectively, and thus modified versions of the proofs of the canonical SC for $G(\mathcal{C})$ and $\Pi(\mathcal{C})$ apply here.

Theorem 10.25. *If $[0, 1]_* = [0, a]_L \oplus [a, 1]_G$, the logic $L_*(\mathcal{C})$ has the canonical SC if, and only if, $a \notin C$.*

Proof: If $a \in C$ we have proved that the logic $L_*(\mathcal{C})$ has not the canonical SC. Now we will prove the canonical standard completeness in the case that $a \notin C$.

The proof is analogous (with adequate changes) to the one given for the expansion of Gödel logic with truth-constants. We will sketch it. We know that the logic enjoys the FSSC and thus we have to prove that tautologies of the canonical standard chain are contained in the tautologies of any other standard chain. The proof is by contraposition. Suppose that there is a formula φ and an evaluation e over a standard chain defined by a proper filter F such that $e(\varphi) < 1$ and we have to prove that there is an evaluation e' over the canonical standard chain such that $e'(\varphi) < 1$.

Take $X = \{e(\psi) \mid \psi \text{ subformula of } \varphi\} \cup \{0, 1\}$ and let $\alpha = \min\{r \in F \mid \bar{r} \text{ appears in } \varphi\}$. Now, define $f : X \rightarrow [0, \alpha]$ by stipulating that its restriction over $X \cap [0, a]$ is the identity function and its restriction over $X \cap [a, 1]$ is an increasing function with $f(a) = a$ and $f(1) = \alpha$.

Then define e' as the evaluation over the canonical standard algebra such that

$$e'(x) = \begin{cases} f(e(x)), & \text{if } x \text{ propositional variable in } \varphi \\ 1, & \text{if } x \text{ propositional variable not in } \varphi \\ r, & \text{if } x = \bar{r} \end{cases}$$

By induction we can prove that for each subformula ψ of φ we have:

- $e'(\psi) \geq \alpha$, if $e(\psi) = 1$
- $a < e'(\psi) < \alpha$, if $a < e(\psi) < 1$
- $e'(\psi) = e(\psi)$, if either $e(\psi) \in [0, a]$ or $e(\psi) = e(x)$ for some x which is a propositional variable or truth-constant appearing in φ .

In particular, from these properties, we see that the evaluation e' over the canonical standard chain is such that $e'(\varphi) < 1$, which ends the proof. \square

Theorem 10.26. *If $[0, 1]_* = [0, a]_L \oplus [a, 1]_\Pi$, the logic $L_*(\mathcal{C})$ has the canonical SC if, and only if, $a \notin C$.*

Proof: The proof is rather analogous (with the adequate changes) to the proof of canonical standard completeness for the expansion of Product logic with truth-constants. If $a \in C$ we have proved (by a counterexample) that

$L_*(C)$ has not the canonical SC. We will prove that if $a \notin C$, then $L_*(C)$ has the canonical SC. In such a case it is obvious that there are only two proper filters of \mathcal{C} defining two chains over $[0, 1]$: the canonical one (defined by the trivial filter) where each element of C is interpreted as itself, and the chain defined by the filter $F = (a, 1] \cap C$ where each element of C is interpreted as itself if it belongs to the first component and as 1 if it belongs to F .

Take an arbitrary formula $\varphi \in Fm_{\mathcal{L}_C}$ and suppose that $\not\models_{L_*(C)} \varphi$. We want to show that $\not\models_{[0,1]_{L_*(C)}} \varphi$. By the FSSC we know that $\not\models_{\{[0,1]_{L_*(C)}, [0,1]_{L_*(C)}^F\}} \varphi$, so what we have to prove is the following:

If $\not\models_{[0,1]_{L_*(C)}^F} \varphi$, then $\not\models_{[0,1]_{L_*(C)}} \varphi$.

To this end, we first need to prove four claims.

Let the restriction of t-norm $*$ on the interval $[a, 1]$ be defined by

$$u * v = h^{-1}(h(u) \cdot h(v))$$

for some increasing bijection $h : [a, 1] \rightarrow [0, 1]$. Let $t > 0$ and define $k_t : [0, 1] \rightarrow [0, 1]$ by

$$k_t(z) = \begin{cases} z & \text{if } z \in [0, a], \\ h^{-1}((h(z))^t) & \text{otherwise.} \end{cases}$$

Furthermore, for any evaluation e into $[0, 1]_{L_*(C)}^F$ we consider:

(i) e'_t as the evaluation over the canonical standard chain $[0, 1]_{L_*(C)}$ defined for any propositional variable x by,

$$e'_t(x) = k_t(e(x))$$

(ii) e_t^* as the mapping defined by $e_t^*(\varphi) = k_t(e(\varphi))$.

Claim 1. For any formulae φ, ψ ,

(i) $e_t^*(\varphi \& \psi) = e_t^*(\varphi) * e_t^*(\psi)$

(ii) $e_t^*(\varphi \rightarrow \psi) = e_t^*(\varphi) \Rightarrow e_t^*(\psi)$

Proof:

(i.1) If $e_t^*(\varphi \& \psi) > a$ then $e_t^*(\varphi), e_t^*(\psi) > a$, and hence $e(\varphi), e(\psi) > a$ as well. In this case, $e_t^*(\varphi \& \psi) = h^{-1}((h(e(\varphi \& \psi)))^t) = h^{-1}((h(e(\varphi) * e(\psi)))^t) = h^{-1}((h(h^{-1}(h(e(\varphi)) \cdot h(e(\psi))))^t) = h^{-1}((h(e(\varphi)) \cdot h(e(\psi)))^t) = h^{-1}(h(e(\varphi))^t \cdot h(e(\psi))^t) = e_t^*(\varphi) * e_t^*(\psi)$.

(i.2) If $e_t^*(\varphi \& \psi) \leq a$, then $e_t^*(\varphi \& \psi) = e(\varphi \& \psi) = e(\varphi) * e(\psi)$, and hence $e(\varphi) \leq a$ or $e(\psi) \leq a$. W.l.o.g., assume $e(\varphi) = \min(e(\varphi), e(\psi)) \leq a$, and hence $e_t^*(\varphi) = e(\varphi)$. Then, if $e(\psi) > a$ then $e_t^*(\psi) > a$ and $e(\varphi) * e(\psi) = e(\varphi) = e^*(\varphi) * e_t^*(\psi)$. Otherwise, if $e(\psi) \leq a$, then $e_t^*(\psi) = e(\psi)$.

(ii) It follows from (i). □

Claim 2. For any formula ψ ,
 if $e(\psi) \in (a, 1]$, then $e'_t(\psi) \in (a, 1]$,
 if $e(\psi) \in [0, a]$, then $e'_t(\psi) = e(\psi)$.

Proof: The proof is by induction:

- If ψ is a propositional variable, the statement is obviously true by definition of e'_t .
- If ψ is a truth-constant \bar{r} , either $r > a$ and then $e(\bar{r}) = 1$ and $e'_t(\bar{r}) = r > a$, or $r < a$ and then $e(\bar{r}) = r = e'_t(\bar{r})$.
- If $\psi = \delta \& \gamma$, then we have two cases:
 - 1.- If $e(\psi) \in (a, 1]$ then it is so for $e(\delta), e(\gamma)$. and thus for $e'_t(\delta), e'_t(\gamma)$ and, as a consequence, for $e'_t(\psi)$.
 - 2.- If $e(\psi) \in [0, a]$, then at least one of $e(\delta), e(\gamma)$ must belong to $[0, a]$. Suppose that $e(\delta) \in [0, a]$, hence by hypothesis $e'_t(\delta) \in [0, a]$ as well, hence $e'_t(\psi) = e'_t(\delta) * e'_t(\gamma) \leq a$.
- If $\psi = \delta \rightarrow \gamma$, then we have several cases:
 - 1.- If $e(\psi) = 1$, then $e(\delta) \leq e(\gamma)$ and we have two cases:
 - 1.1.- If $e(\delta), e(\gamma)$ belong to the same subinterval the statement is obvious.
 - 1.2.- If $e(\delta), e(\gamma)$ belong to different subintervals, the statement also holds true by the induction hypothesis.
 - 2.- If $e(\psi) < 1$ then $e(\delta) > e(\gamma)$ and we have also two cases:
 - 2.1.- If $e(\psi) > a$, then $e(\delta) > e(\gamma) > a$ and thus $e'_t(\psi) \in (a, 1]$.
 - 2.2.- If $e(\psi) \leq a$, then $e(\delta) > e(\gamma) \in [0, a]$ and we have two possibilities depending on which component $e(\gamma)$ belongs. But, in any case, the induction hypothesis proves easily that $e'_t(\psi) = e(\psi)$.

□

Remark that the set $[0, 1]^{\mathbb{R}^+}$ of all functions from \mathbb{R}^+ into $[0, 1]$ becomes an L_* -algebra with the operations $*$ and \Rightarrow_* defined pointwise and with the constant function 0 as bottom and the constant function 1 as top.

Let $F \subseteq [0, 1]^{\mathbb{R}^+}$ be the set of all functions $f : \mathbb{R}^+ \rightarrow [0, 1]$ satisfying the following condition:

- (E) There exists c such that $a < c \leq 1$ and $t_0 > 0$ such that $c \leq f(t)$ for all $t \geq t_0$.

It is immediate to verify that F is an implicative filter (as defined in [33, Lemma 1.5]) on the L_* -algebra $[0, 1]^{\mathbb{R}^+}$. The congruence relation defined by F on $[0, 1]^{\mathbb{R}^+}$, $f \sim g$ iff $f \Rightarrow g \in F$ and $g \Rightarrow f \in F$, is defined by

$$f \sim g \text{ iff there exist } c, d \in (a, 1] \text{ and } t_0 > 0 \text{ such that} \\ c * g(t) \leq f(t) \leq d \Rightarrow g(t) \text{ for all } t > t_0.$$

Then, one can check that \sim satisfies the following properties, where f_a stands for the constant function with value a .

Claim 3. *The congruence relation \sim satisfies:*

- (i) $f \sim f_a$ if, and only if, there exists t_0 such that $f(t) = a$ for all $t \geq t_0$.
- (ii) Suppose $f \sim g$. Then $\lim_{t \rightarrow \infty} g(t) = a$ if, and only if, $\lim_{t \rightarrow \infty} f(t) = a$.

Proof: Just recall that, if $c, d \in (a, 1]$, then $c * a = d \Rightarrow a = a$. \square

Claim 4. *Let e and e'_t as above be given. For every formula ϕ such that $a < e(\phi) < 1$, let $g_\phi(t) = e_t^*(\phi)$ and $f_\phi(t) = e'_t(\phi)$. Then we have $f_\phi \sim g_\phi$. In particular, $\lim_{t \rightarrow \infty} e'_t(\phi) = a$.*

Proof: Let us proceed by induction on the complexity of ϕ .

1. ϕ is a constant \bar{r} . Then it must be $r > a$, hence $e(\bar{r}) = 1$, and then $g_{\bar{r}}(t) = k_t(e(\bar{r})) = k_t(1) = 1$ and $f_{\bar{r}}(t) = e'_t(\bar{r}) = r$, and obviously $r \sim 1$.
2. ϕ is a propositional variable. Then it is a direct consequence of the definition ($f_x(t) = g_x(t)$).
3. $\phi = (\psi_1 \& \psi_2)$. If $e(\psi_1 \& \psi_2) > a$ then $e(\psi_1), e(\psi_2) > a$, hence $g_{\psi_1}, g_{\psi_2}, f_{\psi_1}, f_{\psi_2} \in F$. Then:
 $g_{\psi_1 \& \psi_2}(t) = e_t^*(\psi_1 \& \psi_2) = e_t^*(\psi_1) * e_t^*(\psi_2) = g_{\psi_1}(t) * g_{\psi_2}(t)$.
 $f_{\psi_1 \& \psi_2}(t) = e'_t(\psi_1 \& \psi_2) = e'_t(\psi_1) * e'_t(\psi_2) = f_{\psi_1}(t) * f_{\psi_2}(t)$.
 Since \sim is a congruence, if we suppose that $f_{\psi_1} \sim g_{\psi_1}$ and $f_{\psi_2} \sim g_{\psi_2}$, we can conclude that $f_{\psi_1 \& \psi_2} \sim g_{\psi_1 \& \psi_2}$.
4. $\phi = (\psi_1 \rightarrow \psi_2)$. If $a < e(\psi_1 \rightarrow \psi_2) < 1$ then $e(\psi_1), e(\psi_2) > a$, hence $g_{\psi_1}, g_{\psi_2}, f_{\psi_1}, f_{\psi_2} \in F$. Then:
 $g_{\psi_1 \rightarrow \psi_2}(t) = e_t^*(\psi_1 \rightarrow \psi_2) = e_t^*(\psi_1) \Rightarrow e_t^*(\psi_2) = g_{\psi_1}(t) \Rightarrow g_{\psi_2}(t)$.
 $f_{\psi_1 \rightarrow \psi_2}(t) = e'_t(\psi_1 \rightarrow \psi_2) = e'_t(\psi_1) \Rightarrow e'_t(\psi_2) = f_{\psi_1}(t) \Rightarrow f_{\psi_2}(t)$.
 Using again the fact that \sim is a congruence, from the hypothesis $f_{\psi_1} \sim g_{\psi_1}$ and $f_{\psi_2} \sim g_{\psi_2}$, we obtain $f_{\psi_1 \rightarrow \psi_2} \sim g_{\psi_1 \rightarrow \psi_2}$.

The first statement of the proposition is proved. The second statement follows from the first statement and (ii) of Claim 3. \square

Now we can obtain the result we are looking for:

Let φ be not valid in $[0, 1]_{L_*(C)}^F$. There exists an evaluation e such that $e(\varphi) < 1$. By last lemma, $\lim_{t \rightarrow \infty} e'_t(\varphi) = a$ as well, hence for some large enough t , $e'_t(\varphi) < 1$. Thus φ is not valid in the canonical standard chain. \square

- (2) If $[0, 1]_*$ is an ordinal sum of three or more components, then $L_*(C)$ has not the canonical SC as the following examples show:

$[0, 1]_*$	Canonical SC for $L_*(C)$
$[0, 1]_L$	Yes
$[0, 1]_G$	Yes
$[0, 1]_\Pi$	Yes
$[0, a]_G \oplus \mathcal{A}$	No
$[0, a]_\Pi \oplus \mathcal{A}$	No
$[0, a]_L \oplus \mathcal{A}, \quad a \in C$	No
$[0, a]_L \oplus [a, 1]_G, \quad a \notin C$	Yes
$[0, a]_L \oplus [a, 1]_\Pi, \quad a \notin C$	Yes
$[0, a]_L \oplus [a, b]_G \oplus \mathcal{A}, \quad a \notin C$	No
$[0, a]_L \oplus [a, b]_\Pi \oplus \mathcal{A}, \quad a \notin C$	No
$[0, 1]_{NM}$	Yes
$[0, 1]_{\otimes_c}$	Yes
$[0, 1]_{\star_c}$	Yes
$[0, 1]_*, \text{ for other } * \in \mathbf{WNM-fin}$	No

Table 10.3: Canonical standard completeness results for logics $L_*(C)$ when $*$ $\in \mathbf{CONT-fin} \cup \mathbf{WNM-fin}$. Recall that \otimes_c and \star_c are those WNM t-norms depicted in Figure 9.3.

2.1.- If $[0, 1]_* = [0, a]_L \oplus [a, b]_G \oplus \mathcal{A}$, take $d \in F = (a, b) \cap C$ in the second component. Then the formula,

$$\bar{d} \rightarrow (\neg\neg x \rightarrow x) \vee (x \rightarrow x \& x)$$

is a tautology of the canonical standard algebra but not of $[0, 1]_{L_*(C)}^F$.

2.2.- If $[0, 1]_* = [0, a]_L \oplus [a, b]_\Pi \oplus \mathcal{A}$, take $d \in F = (a, b) \cap C$ in the second component. Then the formula,

$$\bar{d} \rightarrow (\neg\neg x \& \neg\neg y \& ((x \rightarrow x \& y) \rightarrow y) \& (y \rightarrow x) \& (x \rightarrow x \& x) \rightarrow x)$$

is a tautology of the canonical standard algebra and not of $[0, 1]_{L_*(C)}^F$.

Summarizing (see Table 10.3) the canonical SC holds for the expansion of the logic of a continuous t-norm $*$ which is a finite ordinal sum of the three basic ones by a set of truth-constants if, and only if, $[0, 1]_*$ is either one of the three basic algebras $([0, 1]_L, [0, 1]_G$ or $[0, 1]_\Pi)$ or $[0, 1]_* = [0, a]_L \oplus [a, 1]_\Pi$ or $[0, 1]_* = [0, a]_L \oplus [a, 1]_G$ (with $a \notin C$).

All the results on the canonical SC are gathered in Table 10.3.

10.4 Completeness results for evaluated formulae

This section deals with completeness results when we restrict to what we call *evaluated formulae*, formulae of type $\bar{r} \rightarrow \varphi$, where φ is a formula without

new truth-constants (different from 0 and 1). These formulae can be seen as a special kind of Novák's *evaluated formulae*, which are expressions a/A where a is a truth value (from a given algebra) and A is a formula that may contain truth-constants again, and whose interpretation is that the truth-value of A is at least a . Hence our formulae $\bar{r} \rightarrow \varphi$ would be expressed as r/φ in Novák's evaluated syntax. On the other hand, formulae $\bar{r} \rightarrow \varphi$ when φ is a Horn-like rule of the form $b_1 \& \dots \& b_n \rightarrow h$ also correspond to typical fuzzy logic programming rules $(b_1 \& \dots \& b_n \rightarrow h, r)$, where r specifies a lower bound for the validity of the rule.

From the previous sections we know that the FSSC is true for the expansion of L_* with a suitable subalgebra of truth-constants (not only for evaluated formulae), but the canonical FSSC is only true for expansions of Lukasiewicz logic. Restricting the language to evaluated formulae these results can be improved. To describe them we divide the subject by cases.

10.4.1 The case of continuous t-norms

Next theorems state the canonical FSSC restricted to evaluated formulae for the expansions of Gödel and Product logics with truth-constants.

Lemma 10.27. *Let $a \in (0, 1]$ and define a mapping $f_a : [0, 1] \rightarrow [0, 1]$ as follows:*

$$f_a(x) = \begin{cases} 1, & \text{if } x \geq a \\ x, & \text{otherwise} \end{cases}$$

Then f_a is a homomorphism with respect to the standard Gödel truth functions. Therefore, if e is a evaluation of the formulae, then $e_a = f_a \circ e$ is another evaluation.

Proof: We have to prove: (i) $f_a(0) = 0$, (ii) $f_a(\min(x, y)) = \min(f_a(x), f_a(y))$, and (iii) $f_a(x \Rightarrow_G y) = f_a(x) \Rightarrow_G f_a(y)$. (i) is obvious and (ii) is also easy immediate since f_a is a non-decreasing function. So let us prove (iii). We consider two cases:

Case A : $x \leq y$, $x \Rightarrow_G y = 1$. In this case, $f_a(x) \leq f_a(y)$ as well, hence $f_a(x \Rightarrow_G y) = f(1) = 1 = f_a(x) \Rightarrow_G f_a(y)$.

Case B : $x > y$, $x \Rightarrow_G y = y$. Now we distinguish the following three sub-cases:

B.1 : $a \leq y < x$, $f_a(x \Rightarrow_G y) = 1$. In this case $f_a(x) = f_a(y) = 1$ and hence $f_a(x) \Rightarrow_G f_a(y) = 1$;

B.2 : $y < a \leq x$, $f_a(x \Rightarrow_G y) = y$. In this case $f_a(x) = 1$, $f_a(y) = y$ and hence $f_a(x) \Rightarrow_G f_a(y) = y$;

B.3 : $y < x < a$, $f_a(x \Rightarrow_G y) = y$. In this case $f_a(y) = y$, $f_a(x) = x$, and hence $f_a(x) \Rightarrow_G f_a(y) = y$.

So, in any of the subcases, $f_a(x \Rightarrow_G y) = f_a(x) \Rightarrow_G f_a(y)$.

This ends the proof. \square

Theorem 10.28. $G(\mathcal{C})$ has the canonical FSSC if we restrict the language to evaluated formulae, i.e. for any finite index set I we have:

$$\{\bar{r}_i \rightarrow \varphi_i\}_{i \in I} \vdash_{G(\mathcal{C})} \bar{s} \rightarrow \psi \text{ iff } \{\bar{r}_i \rightarrow \varphi_i\}_{i \in I} \models_{[0,1]_{G(\mathcal{C})}} \bar{s} \rightarrow \psi.$$

where $\psi, \varphi_i \in Fm_{\mathcal{L}}$.

Proof: Suppose that $I = \{1, \dots, n\}$. One direction is easy. As for the difficult one, by the Deduction-detachment Theorem, it is enough to prove that if there is an evaluation e which is not a model of $(\bigwedge_{i=1}^n (\bar{r}_i \rightarrow \varphi_i) \rightarrow (\bar{s} \rightarrow \psi))$, then we can find another evaluation e' which is model of $\{\bar{r}_i \rightarrow \varphi_i\}_{i \in I}$ and not of $\bar{s} \rightarrow \psi$.

So let e be such that $e(\bigwedge_{i=1}^n (\bar{r}_i \rightarrow \varphi_i) \rightarrow (\bar{s} \rightarrow \psi)) < 1$. If e is a model of every $\bar{r}_i \rightarrow \varphi_i$, then we can take $e' = e$ and the problem is solved. Otherwise, there exists some $1 \leq j \leq n$ for which $r_j > e(\varphi_j)$ and thus $e(\bar{r}_j \rightarrow \varphi_j) = e(\varphi_j) < 1$. Let $J = \{j \mid r_j > e(\varphi_j)\}$ and let $a = e(\bigwedge_{i=1}^n \bar{r}_i \rightarrow \varphi_i) = \min\{e(\varphi_j) \mid j \in J\}$. Then the evaluation e' such that $e' = e_a$ over the propositional variables does the job. Namely, by Lemma 10.27, over Gödel formulae we have $e' = e_a \geq e$, so e' is still a model of $\bar{r}_i \rightarrow \varphi_i$ for every $i \in \{1, \dots, n\} \setminus J$. But now, $e'(\varphi_j) = 1$ for every $j \in J$, so e' is also a model of $\{\bar{r}_i \rightarrow \varphi_i\}_{i \in I}$. On the other hand, since $e(\bigwedge_{i=1}^n (\bar{r}_i \rightarrow \varphi_i) \rightarrow (\bar{s} \rightarrow \psi)) < 1$, it must be $s > e(\psi)$ and $a = e(\bigwedge_{i=1}^n (\bar{r}_i \rightarrow \varphi_i)) > e(\psi)$. Now, by Lemma 10.27, $e'(\psi) = e_a(\psi) = e(\psi)$, hence $e'(\bar{s} \rightarrow \psi) = e(\bar{s} \rightarrow \psi) < 1$. Therefore we have proved the theorem. \square

Theorem 10.29. $\Pi(\mathcal{C})$ has the canonical FSSC if we restrict the language to evaluated formulae, i.e. for any finite index set I we have:

$$\{\bar{r}_i \rightarrow \varphi_i\}_{i \in I} \vdash_{\Pi(\mathcal{C})} \bar{s} \rightarrow \psi \text{ iff } \{\bar{r}_i \rightarrow \varphi_i\}_{i \in I} \models_{[0,1]_{\Pi(\mathcal{C})}} \bar{s} \rightarrow \psi.$$

where $\psi, \varphi_i \in Fm_{\mathcal{L}}$.

Proof: Actually, as always, one direction (soundness) is easy due to the book-keeping axioms. To prove the converse direction it is enough to combine the FSSC with the following result.

Claim: If $\{\bar{r}_i \rightarrow \varphi_i \mid i \in I\} \models_{[0,1]_{\Pi(\mathcal{C})}} \bar{s} \rightarrow \psi$ then $\{\bar{r}_i \rightarrow \varphi_i\}_{i \in I} \models_{[0,1]_{\Pi(\mathcal{C})}^*} \bar{s} \rightarrow \psi$

Proof: Without loss of generality we may assume $r_i > 0$ for all i and $s > 0$. Suppose $\{\bar{r}_i \rightarrow \varphi_i\}_{i \in I} \not\models_{[0,1]_{\Pi(\mathcal{C})}^*} \bar{s} \rightarrow \psi$. Assume also $I = \{1, \dots, n\}$. Then there exists a $[0, 1]_{\Pi(\mathcal{C})}^*$ -evaluation e such that $e(\bar{r}_1 \rightarrow \varphi_1) = \dots = e(\bar{r}_n \rightarrow \varphi_n) = 1$ and $e(\bar{s} \rightarrow \psi) < 1$. Since $e(r_i) = e(s) = 1$ for all i , we also have $e(\varphi_1) = \dots = e(\varphi_n) = 1$ and $e(\psi) < 1$.

Assume $e(\psi) = 0$. Then, letting e' be the $[0, 1]_{\Pi(\mathcal{C})}$ -evaluation defined by $e'(p) = e(p)$ for any propositional variable p , we have $e'(\bar{r}_1 \rightarrow \varphi_1) = \dots = e'(\bar{r}_n \rightarrow \varphi_n) = 1$ and $e'(\psi) = 0$, hence $\{\bar{r}_1 \rightarrow \varphi_1, \dots, \bar{r}_n \rightarrow \varphi_n\} \not\models_{[0,1]_{\Pi(\mathcal{C})}} \bar{s} \rightarrow \psi$.

Assume $e(\psi) > 0$. Let $\alpha \in \mathbb{R}^+$ such that $(e(\psi))^\alpha < s$. Then the $[0, 1]_{\Pi(\mathcal{C})}$ -evaluation e' , where $e'(p) = (e(p))^\alpha$ for any propositional variable p , is such that $e'(\bar{r}_i \rightarrow \varphi_i) = 1$ for all i but $e'(\bar{s} \rightarrow \psi) < 1$, hence $\{\bar{r}_1 \rightarrow \varphi_1, \dots, \bar{r}_n \rightarrow \varphi_n\} \not\models_{[0,1]_{\Pi(\mathcal{C})}} \bar{s} \rightarrow \psi$. \square

Finally, as a direct consequence of the FSSC and the above claim we obtain the desired result. \square

Now, we will study the canonical SC and the canonical FSSC restricted to evaluated formulae for other logics. Take any $*$ \in **CONT-fin** which is an ordinal sum of more than one basic component and suppose that the first component is defined on the interval $[0, a]$. In the following cases we can refute the canonical SC (and hence the canonical FSSC as well):

1. The first component of the t-norm $*$ is a copy of Łukasiewicz t-norm and $a \in C$.
2. The first component of the t-norm $*$ is a copy of product t-norm.
3. The first component of the t-norm $*$ is a copy of minimum t-norm.
4. There are more than two components and the second component is a copy of minimum t-norm.
5. There are more than two components and the second component is a copy of product t-norm.

Indeed, for all these cases we can use the same counterexample that was given in the previous section to show that the corresponding logics do not enjoy the canonical SC, because the counterexamples were actually evaluated formulae.

The following theorem deals with the remaining case of ordinal sums of two basic components. The case $[0, 1]_* = [0, a]_L \oplus [a, 1]_L$ is not considered here since in such a situation, under the working hypothesis that there exists $b \in (a, 1]$ such that $b \in C$, necessarily $a \in C$ as well.

Theorem 10.30. *The restriction to evaluated formulae of the logic $L_*(C)$ when either $[0, 1]_* = [0, a]_L \oplus [a, 1]_G$ or $[0, 1]_* = [0, a]_L \oplus [a, 1]_\Pi$, and $a \notin C$ has the canonical FSSC.*

Proof: The proof is an easy modification of the proofs given for $G(C)$ and $\Pi(C)$. Here we only sketch the proof for $[0, 1]_* = [0, a]_L \oplus [a, 1]_\Pi$. What we want to prove is:

$$\begin{aligned} \{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \vdash_{L_*(C)} \bar{s} \rightarrow \psi \\ \text{if, and only if,} \\ \{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \models_{[0,1]_{L_*(C)}} \bar{s} \rightarrow \psi \end{aligned}$$

where φ_i and ψ are $L_*(C)$ -formulae, i.e., formulae not containing truth-constants different from $\bar{0}$ and $\bar{1}$. Actually, as always, one direction (soundness) is obvious. To prove the converse direction

$$\begin{aligned} \text{If } \{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \models_{[0,1]_{L_*(C)}} \bar{s} \rightarrow \psi, \text{ then} \\ \{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \vdash_{L_*(C)} \bar{s} \rightarrow \psi \end{aligned}$$

it is enough to combine the FSSC of $L_*(C)$ with the following result:

Claim 5. *If $\{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \models_{[0,1]_{L_*(C)}} \bar{s} \rightarrow \psi$ then $\{\bar{r}_1 \rightarrow \varphi_1, \dots, \bar{r}_n \rightarrow \varphi_n\} \models_{[0,1]_{L_*(C)}^F} \bar{s} \rightarrow \psi$ being $F = (a, 1] \cap C$.*

To prove it and without loss of generality we may assume $r_i > 0$ for all i and $s > 0$. Suppose $\{\bar{r}_1 \rightarrow \varphi_1, \dots, \bar{r}_n \rightarrow \varphi_n\} \not\models_{[0,1]_{L_*(C)}^F} \bar{s} \rightarrow \psi$. Then there exists a $[0, 1]_{L_*(C)}^F$ -evaluation e such that $e(\bar{r}_1 \rightarrow \varphi_1) = \dots = e(\bar{r}_n \rightarrow \varphi_n) = 1$ and $e(\bar{s} \rightarrow \psi) < 1$.

(i) If $s \in (0, a]$, and hence $e(\bar{s}) = s$ and $e(\psi) < s$, then take the evaluation e' over the canonical standard chain defined by $e'(p) = e(p)$ for any propositional variable p . Notice that, since $e(\bar{r}) \geq e'(\bar{r})$ and $e(\varphi) = e'(\varphi)$, it is easy to compute that $e'(\bar{r}_1 \rightarrow \varphi_1) = \dots = e'(\bar{r}_n \rightarrow \varphi_n) = 1$ and $e'(\bar{s} \rightarrow \psi) = e(\bar{s} \rightarrow \psi) < 1$.

(ii) If $s \in (a, 1]$, and hence $e(\bar{s}) = 1$ and $e(\psi) < 1$, we can assume $e(\psi) \geq s$, otherwise the above evaluation e' does the job. Then take the family of evaluations e'_t over the canonical standard chain defined by $e'_t(p) = k_t(e(p))$ for any propositional variable p , where $k_t : [0, 1] \rightarrow [0, 1]$ is the mapping defined in the proof of Theorem 10.26, i.e.

$$k_t(z) = \begin{cases} z & \text{if } z \in [0, a], \\ h^{-1}((h(z))^t) & \text{otherwise.} \end{cases}$$

By definition of k_t it is easy to find a large enough t such that $a < e'_t(\psi) < s$, and hence $e'_t(\bar{s} \rightarrow \psi) < 1$. Moreover, it is easy to check that we still have $e'_t(\bar{r}_1 \rightarrow \varphi_1) = \dots = e'_t(\bar{r}_n \rightarrow \varphi_n) = 1$. Indeed, if $r_i \in (a, 1]$, then $e(r_i) = 1$ and $e(\varphi) = 1$, hence $e'_t(\varphi) = 1$ as well. If $r_i \in (0, a]$, then $e'_t(\bar{r}_i) = e(\bar{r}_i) = r_i$ and $e(\varphi_i) \geq r_i$. Now, if $e(\varphi_i) \leq a$ then $e'_t(\varphi_i) = e(\varphi_i)$, otherwise, if $e(\varphi_i) > a$ then $e'_t(\varphi_i) > a$ as well. In any case, $e'_t(\varphi_i) \geq r_i$, hence $e'_t(\bar{r}_i \rightarrow \varphi_i) = 1$. \square

All these results are summarized in Table 10.4, where interestingly enough it turns out that both standard completeness properties (SC and FSSC) restricted to evaluated formulae are equivalent, for each $* \in \mathbf{CONT-fin}$.

10.4.2 The case of WNM-fin t-norms

For t-norms from **WNM-fin** we must restrict to evaluated formulae of the kind $\bar{r} \rightarrow \varphi$ where r is a positive constant. We will call them *positively evaluated formulae*. The next example shows that the restriction to this kind of evaluated formulae is indeed necessary.

Example 8. *Let $*$ be a WNM t-norm different from the minimum t-norm and let C be a countable subalgebra of $[0, 1]_*$ such that $C_- \setminus \{0\} \neq \emptyset$. Take any $r \in C_- \setminus \{0\}$. Then, we have:*

$$\bar{r} \rightarrow \neg(x \rightarrow y) \models_{[0,1]_{L_*(C)}} y \rightarrow x$$

and

$$\bar{r} \rightarrow \neg(x \rightarrow y) \not\models_{[0,1]_{L_*(C)}^{F \neg^r}} y \rightarrow x$$

hence, $L_*(C)$ does not enjoy the canonical FSSC restricted to evaluated formulae when we allow negative constants.

We obtain a positive result in the next theorem and several negative results as a consequence of the examples given in the subsection 10.3.2. The positive one corresponds to the three families of t-norms in Figure 9.3 and it is proved analogously to the case of $G(C)$, using now this analogous lemma:

Lemma 10.31. *For every $*$ $\in \{\otimes, \star, \odot\}$, let $a \in (c, 1]$ and define a mapping $f^a : [0, 1] \rightarrow [0, 1]$ as follows:*

$$f^a(x) = \begin{cases} 1, & \text{if } x \geq a \\ 0, & \text{if } x \leq n_{*c}(a) \\ x, & \text{otherwise} \end{cases}$$

Then f^a is a morphism with respect to the operations of the algebra $[0, 1]_{+c}$. Therefore, if e is a evaluation on $[0, 1]_{L_{*c}(C)}$, then $e^a = f^a \circ e$ is another evaluation on $[0, 1]_{L_{*c}(C)}$.

Theorem 10.32. *If $*$ is one of the three WNM t-norms depicted in Figure 9.3 then $L_*(C)$ has the canonical FSSC if we restrict the language to evaluated formulae. More precisely, given a finite index set I , we have:*

$$\{\bar{r}_i \rightarrow \varphi_i\}_{i \in I} \vdash_{L_*(C)} \bar{s} \rightarrow \psi \text{ iff } \{\bar{r}_i \rightarrow \varphi_i\}_{i \in I} \models_{[0, 1]_{L_{*c}(C)}} \bar{s} \rightarrow \psi.$$

where $\psi, \varphi_i \in Fm_{\mathcal{L}}$ and $r_i \in (c, 1]$.

Notice that in all these logics, the positive constants coincide with the interval $(c, 1] \cap C$, except for the logics corresponding to \odot_c with $c > 1/2$ where the positive constants are those in $(1 - c, 1] \cap C$. The case $L_* = NM$ appears above when $*$ $= \star_{1/2} = \odot_{1/2}$, and then the condition for the constants is $r_i \in (\frac{1}{2}, 1]$.

For $*$ $\in \mathbf{WNM-fin}$ other than \otimes_c and \star_c the canonical FSSC restricted to positively evaluated formulae does not hold as the following counterexamples show.

Example 9. *Let $*$ $= \odot_c$ with $c > 1/2$. Let $r \in C$ such that $1 - c < r \leq c$. Then the semantical deduction*

$$\neg\neg p(x) \rightarrow p(x) \models \bar{r} \rightarrow p(x)$$

is valid in $[0, 1]_{L_*(C)}$ but not in $[0, 1]_{L_{*c}(C)}^F$ for any F containing r . Obviously, in $[0, 1]_{L_{*c}(C)}$ any involutive and positive element is greater than r .

Example 10. *Let $*$ $\in \mathbf{WNM-fin}$ be such that the first interval I of the partition associated to $*$ formed by positive elements is involutive and there is a constant interval on the right of it. In such a case, take a truth-constant r in the interior of I . Then the semantical deduction,*

$$(\neg\neg p(x) \rightarrow p(x)) \rightarrow p(x) \models \bar{r} \rightarrow p(x)$$

is valid in $[0, 1]_{L_*(C)}$ but not in $[0, 1]_{L_{*c}(C)}^F$ for any F containing r . Observe that in $[0, 1]_{L_{*c}(C)}$ the premise is true if, and only if, $p(x)$ is not involutive or 1, and for these cases $p(x)$ is greater than r .

Example 11. Let $* \in \mathbf{WNM-fin}$ such that the first interval of the partition associated to $*$ formed by positive elements is a constant interval with respect to the negation (I_c being c the biggest element of the interval). Additionally suppose that there is another interval of positive elements that is also a constant interval with respect to the negation. In such a case, take a truth-constant $r \in I_c$. Then the formula,

$$\bar{r} \rightarrow \neg\neg p(x)$$

is a tautology for $[0, 1]_{L_*(C)}$ and it is not a tautology for $[0, 1]_{L_*(C)}^F$ for any F containing r . Obviously in $[0, 1]_{L_*(C)}$ any involutive and positive element is greater than r .

Example 12. Let $* \in \mathbf{WNM-fin}$ be such that there is a positive element which is a discontinuity point of the negation function. Then, due to symmetry of negation functions, there is a constant interval whose elements are negative and whose greatest element is not the fixpoint. Denote by I the greatest constant interval formed by negative elements whose greatest element is different from the fixpoint and take r as the greatest element of I , i.e. $I = I_r$. Then the semantical deduction,

$$\left\{ \begin{array}{l} \neg\neg n(x) \rightarrow \neg(\neg\neg n(x) \rightarrow n(x)), \\ \neg n(x) \rightarrow \neg(\neg n(x) \rightarrow \neg\neg n(x)) \end{array} \right\} \models \neg\bar{r} \rightarrow \neg n(x)$$

is valid deduction in $[0, 1]_{L_*(C)}$ but it is not in $[0, 1]_{L_*(C)}^F$ for any F containing r . Observe that the first premise is true if, and only if, $n(x)$ is either not involutive or $n(x) = 0$ and the second premise is true if and only if $n(x)$ does not belong to a constant interval whose greatest element is the fixpoint. Thus, if x satisfies the premises, it is clear that $n(x)$ belongs to a constant interval which does not contain the fixpoint, thus it is less or equal to r , and hence the conclusion is also satisfied.

This four examples, as in the case of general SC studied in the last section, prove that a rather large family of expansions of the logic of a WNM t-norm with truth constants do not enjoy canonical FSSC even when we restrict the language to positively evaluated formulae.

The reader can see a summary of all these completeness results in Table 10.4. Notice that the canonical SC restricted to positively evaluated formulae remains an open problem when $* \in \mathbf{WNM-fin}$ is not one the t-norms \otimes_c or \star_c in Figure 9.3. In fact, in the cases considered in Example 11, the canonical SC does not hold, but we still do not know whether it is true in other cases.

Furthermore, comparing this table with Table 10.3 we realise that for a logic $L_*(C)$ where $* \in \mathbf{CONT-fin} \cup \mathbf{WNM-fin}$ (except for the case which remains open), the canonical SC turns out to be equivalent to the canonical SC (and to the canonical FSSC) restricted to positively evaluated formulae.

Open problem: Are these equivalencies valid for wider classes of $L_*(C)$ logics?

Restricted to pos. evaluated formulae of $L_*(\mathcal{C})$		
$[0, 1]_*$	Canonical SC	Canonical FSSC
$[0, 1]_{\mathbf{L}}$	Yes	Yes
$[0, 1]_{\mathbf{G}}$	Yes	Yes
$[0, 1]_{\mathbf{H}}$	Yes	Yes
$[0, a]_{\mathbf{G}} \oplus \mathcal{A}$	No	No
$[0, a]_{\mathbf{H}} \oplus \mathcal{A}$	No	No
$[0, a]_{\mathbf{L}} \oplus \mathcal{A}, \quad a \in C$	No	No
$[0, a]_{\mathbf{L}} \oplus [a, 1]_{\mathbf{G}}, \quad a \notin C$	Yes	Yes
$[0, a]_{\mathbf{L}} \oplus [a, 1]_{\mathbf{H}}, \quad a \notin C$	Yes	Yes
$[0, a]_{\mathbf{L}} \oplus [a, b]_{\mathbf{G}} \oplus \mathcal{A}, \quad a \notin C$	No	No
$[0, a]_{\mathbf{L}} \oplus [a, b]_{\mathbf{H}} \oplus \mathcal{A}, \quad a \notin C$	No	No
$[0, 1]_{\mathbf{NM}}$	Yes	Yes
$[0, 1]_{\otimes_c}$	Yes	Yes
$[0, 1]_{*_c}$	Yes	Yes
$[0, 1]_*, \text{ for other } * \in \mathbf{WNM-fin}$?	No

Table 10.4: Canonical SC and FSSC results restricted to positively evaluated formulae for logics $L_*(\mathcal{C})$ when $* \in \mathbf{CONT-fin} \cup \mathbf{WNM-fin}$.

10.5 Adding truth-constants to expansions with Δ connective

Some algebraizable expansions of MTL have been introduced in the literature. Among them, a remarkable set of expansions are those obtained by enriching the language with the projection connective Δ (see [7]). Namely, given any axiomatic extension L of MTL, the expansion L_Δ is defined by adding to the language a unary connective Δ , and adding to the Hilbert-style system of L the following axiom schemata:

$$(\Delta 1) \quad \Delta\varphi \vee \neg\Delta\varphi$$

$$(\Delta 2) \quad \Delta(\varphi \vee \psi) \rightarrow (\Delta\varphi \vee \Delta\psi)$$

$$(\Delta 3) \quad \Delta\varphi \rightarrow \varphi$$

$$(\Delta 4) \quad \Delta\varphi \rightarrow \Delta\Delta\varphi$$

$$(\Delta 5) \quad \Delta(\varphi \rightarrow \psi) \rightarrow (\Delta\varphi \rightarrow \Delta\psi)$$

and the rule of necessitation:

$$\frac{\varphi}{\Delta\varphi}$$

This logic is algebraizable and its equivalent algebraic semantics is the variety of L_Δ -algebras, i. e. expansions with Δ of L -algebras satisfying the translation

of the axioms $(\Delta 1), \dots, (\Delta 5)$ and the equation $\Delta \bar{1} \approx \bar{1}$. It is easy to prove that all L_Δ -algebras are representable as subdirect products of L_Δ -chains. The interpretation of the Δ connective in these chains is very simple, namely if \mathcal{A} is an L_Δ -chain, then $\Delta^{\mathcal{A}}(\bar{1}^{\mathcal{A}}) = \bar{1}^{\mathcal{A}}$ and $\Delta^{\mathcal{A}}(a) = \bar{0}^{\mathcal{A}}$ for every $a \in A \setminus \{\bar{1}^{\mathcal{A}}\}$.

Proposition 10.33. *For every axiomatic extension L of MTL, L_Δ is a conservative expansion of L .*

Proof: It is obvious that every L -chain is the reduct of an L_Δ -chain (just take the same chain and define Δ in the only possible way for chains), thus we can apply Proposition 2.18. \square

Since there is a one-to-one correspondence between L -chains and L_Δ -chains, we obtain the following result for the SSC and the FSSC of these logics.

Theorem 10.34. *For every algebraizable expansion L of MTL, we have: L has the SSC (resp. FSSC) with respect to a class of standard L -chains \mathbb{K} if, and only if, L_Δ has the SSC (resp. FSSC) with respect to the class of standard L_Δ -chains \mathbb{K}_Δ , where \mathbb{K}_Δ denotes the class of Δ -expansions of chains in \mathbb{K} .*

Proof: It is an easy consequence of Theorems 5.3 and 5.2. \square

Now we will consider expansions with truth-constants for logics with Δ . Given a left-continuous t -norm $*$ and a countable subalgebra $\mathcal{C} \subseteq [0, 1]_*$, we define the logic $L_{*\Delta}(\mathcal{C})$ as the expansion of $L_{*\Delta}$ in the language $\mathcal{L}_{\mathcal{C}}$ obtained by adding the following book-keeping axioms:

$$\begin{aligned} \bar{r} \&\bar{s} &\leftrightarrow \bar{r} * \bar{s} \\ (\bar{r} \rightarrow \bar{s}) &\leftrightarrow \bar{r} \Rightarrow \bar{s} \\ \Delta \bar{r} &\leftrightarrow \Delta \bar{r} \end{aligned}$$

for every $r, s \in \mathcal{C}$.

Again, using the general facts mentioned in the preliminaries we know that $L_{*\Delta}(\mathcal{C})$ is an algebraizable logic and we can axiomatize its equivalent algebraic semantics, the variety of $L_{*\Delta}(\mathcal{C})$ -algebras. Moreover, it can be easily checked that $L_{*\Delta}(\mathcal{C})$ -algebras are representable as subdirect product of chains.

Proposition 10.35. *For every left-continuous t -norm $*$ and every countable subalgebra $\mathcal{C} \subseteq [0, 1]_*$, the logic $L_{*\Delta}(\mathcal{C})$ is a conservative expansion of $L_{*\Delta}$, whenever $L_{*\Delta}$ has the FSSC.*

Proof: Let us denote by \mathbb{S} is the class of standard $L_{*\Delta}$ -chains and by $\mathbb{S}(\mathcal{C})$ is the class of standard $L_{*\Delta}(\mathcal{C})$ -chains. Let $\Gamma \cup \{\varphi\}$ be arbitrary formulae of $L_{*\Delta}$ and suppose that $\Gamma \vdash_{L_{*\Delta}(\mathcal{C})} \varphi$. Then, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{L_{*\Delta}(\mathcal{C})} \varphi$, and this implies that $\Gamma_0 \models_{\mathbb{S}(\mathcal{C})} \varphi$. Since the new truth-constants do not occur in $\Gamma_0 \cup \{\varphi\}$, we have $\Gamma_0 \models_{\mathbb{S}} \varphi$, and by FSSC of $L_{*\Delta}$, $\Gamma_0 \vdash_{L_{*\Delta}} \varphi$, and hence $\Gamma \vdash_{L_{*\Delta}} \varphi$. \square

Hence, for all $* \in \mathbf{CONT} \cup \mathbf{WNM}$, $L_{*\Delta}(\mathcal{C})$ is a conservative expansion of $L_{*\Delta}$.

Since $L_{*\Delta}$ -chains are simple, adding Δ to $L_*(\mathcal{C})$ -chains simplifies significantly their structure as next lemma shows.

Lemma 10.36. *Let \mathcal{A} be a non-trivial $L_{*\Delta}(\mathcal{C})$ -chain, $*$ be a left-continuous t -norm and $\mathcal{C} \subseteq [0, 1]_*$ be a countable subalgebra. Then, for every $r, s \in \mathcal{C}$ such that $r < s$, we have $\bar{r}^{\mathcal{A}} < \bar{s}^{\mathcal{A}}$.*

Proof: Suppose $\bar{r}^{\mathcal{A}} = \bar{s}^{\mathcal{A}}$. Then, $\bar{1}^{\mathcal{A}} = \Delta \bar{1}^{\mathcal{A}} = \Delta \bar{s} \rightarrow \bar{r}^{\mathcal{A}} = \overline{\Delta(s \rightarrow t)}^{\mathcal{A}} = \bar{0}^{\mathcal{A}}$; a contradiction. \square

Therefore, in the variety of $L_{*\Delta}(\mathcal{C})$ -algebras there is only one (up to isomorphism) standard chain over $[0, 1]_*$, the canonical one that we denote by $[0, 1]_{L_{*\Delta}(\mathcal{C})}$. This has several nice consequences, which generalize the results for the continuous case given in [50].

Theorem 10.37. *Let $* \in \mathbf{CONT-fin} \cup \mathbf{WNM-fin}$ and let $\mathcal{C} \subseteq [0, 1]_*$ be a suitable countable subalgebra. Then:*

1. $L_{*\Delta}(\mathcal{C})$ has the canonical FSSC.
2. $L_{*\Delta}(\mathcal{C})$ is a conservative expansion of $L_*(\mathcal{C})$ iff $L_*(\mathcal{C})$ has the canonical FSSC, i.e. iff $*$ is Lukasiewicz t -norm.
3. $L_{*\Delta}(\mathcal{C})$ has the canonical SSC iff $* \in \mathbf{WNM-fin}$.

In Figure 10.1 we show which of the considered expansions of L_* are always conservative (the ones represented by bold arrows).

10.6 Conclusions

In this chapter we have focused on an algebraic approach to study expansions of propositional logics of a left-continuous t -norm with truth-constants. Specially, we have surveyed in detail completeness results for the expansions of logics of left-continuous t -norms with a set of truth-constants $\{\bar{r} \mid r \in \mathcal{C}\}$, for a suitable countable $\mathcal{C} \subseteq [0, 1]$, when (i) the t -norm is a finite ordinal sum of basic components or is WNM t -norm with finite partition, and (ii) the set of truth-constants *covers* all the unit interval in the sense that the interior of each basic component of the t -norm (in the case of continuous t -norms) or of each interval of the partition (in the case of the WNM t -norms) contains at least one value of \mathcal{C} . From a practical point of view, this latter condition seems to correspond to the most interesting case for fuzzy logic-based systems, since they usually consider a set of truth values spread all over the real unit interval, and hence it is natural to assume there are elements of \mathcal{C} in each component or partition of the t -norm.

Of course a lot of expansions with truth-constants remain to be studied, among them:

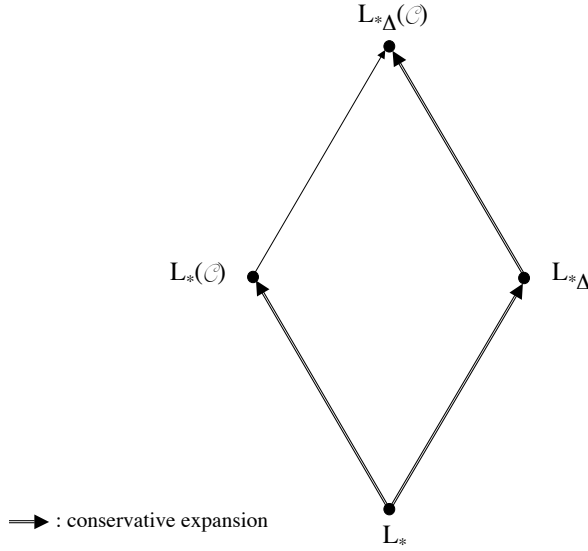


Figure 10.1: Diagram of expansions for $*$ $\in \mathbf{CONT}\text{-}\mathbf{fin} \cup \mathbf{WNM}\text{-}\mathbf{fin}$.

- the case of a logic of a t-norm with a Łukasiewicz component containing some $r \in C$ which generates an infinite MV-chain (in other words, when r corresponds to an irrational value in the isomorphic copy of the component over $[0, 1]$);
- the case when the set of truth-constants does not cover the unit interval;
- the case of continuous t-norms which are the ordinal sum of infinitely many components;
- the case of any other left-continuous t-norm, in particular WNM t-norms with infinite partition.

It seems that for the cases when either the t-norm has infinite components or the set C does not cover $[0, 1]$, a methodology similar to the one used in this chapter could be applied. But in fact there is an explosion of cases to be considered and the need of new definitions and tools seems unavoidable. Let us show a couple of illustrative examples, the first when the set C does not cover $[0, 1]$ and the second when the t-norm has infinite components.

Example 1. Let $[0, 1]_* = [0, a]_{\Pi} \oplus [a, 1]_{\Pi}$ and let $C = \{0, 1\} \cup \{b^n \mid n \in \mathbb{N}\}$ for some $b < a$. Obviously, there are only two proper filters of C , $F_1 = \{1\}$ and $F_2 = C \setminus \{0\}$ but there are (up to isomorphism) three standard $L_*(C)$ -chains.

One, of type F_2 in the sense used in this chapter, is the $L_*(\mathcal{C})$ -chain over $[0, 1]_*$ where the constants different from $\bar{0}$ are interpreted as 1 and $\bar{0}$ is interpreted as 0. The other two are of type F_1 . They are both $L_*(\mathcal{C})$ -chains over $[0, 1]_*$ where all constants are interpreted as different elements, either as powers of an element of the first product component or as powers of an element of the second product component. Of course, these two algebras are not isomorphic. This example shows that in general there is not a bijection between proper filters and standard algebras and, even though it seems possible to have the partial embedding property, the notion and treatment of standard chains should be modified in the case that C does not cover all components.

Example 2. Let $[0, 1]_* = \bigoplus_{n \in \mathbb{N}} [a_n, a_{n+1}]_{\mathbb{L}}$, where $a_n = n/(n+1)$, be an infinite ordinal sum of Łukasiewicz components where the idempotent elements form an increasing sequence with limit 1. For a given $k > 2$, let C_i the carrier of the k -element MV-subalgebra of $[a_i, a_{i+1}]_{\mathbb{L}}$ and denote its elements as $r_{1i} = a_i, r_{2i}, \dots, r_{ki} = a_{i+1}$. Take $C = \bigcup_{i \in \mathbb{N}} C_i \cup \{1\}$. It is clear that C covers all the components but there are standard algebras where the interpretations of the truth-constants do not cover all the components. Indeed, let f be any strictly increasing mapping $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(1) = 1$. One standard $L_*(\mathcal{C})$ -algebra is the chain over $[0, 1]_*$ where $\overline{r_{ij}}$ is interpreted as $r_{f(i)j}$. An easy computation shows that this interpretation defines a standard $L_*(\mathcal{C})$ -chain where the interpretations of truth-constants do not cover the real unit interval. In fact, if $f(i+1)$ is not the successor of $f(i)$ (there are some natural numbers in between), the corresponding components contain no interpretations of truth-constants.

We conjecture that the study of completeness results for the expansions of the remaining logics of continuous t-norms, like the ones in the above examples, will be more in a case-by-case basis rather than by means of a new general theory. Another important issue to be addressed is the predicate calculi of these expanded logics. All these issues are matters for future research.

Chapter 11

Final conclusions and open problems

In this dissertation we have carried out an attempt to describe the axiomatic extensions of the basic t-norm based logic, MTL. We have done it from an algebraic point of view, by exploiting the fact that these logics are algebraizable by varieties of MTL-algebras. Therefore, our study has resulted in an algebraic study of subvarieties of MTL, where the final aim would be to obtain a description of the structure of the lattice of these subvarieties and their relevant properties. Although this description has not been achieved yet, we have done several significant advances in this direction that can be classified in two groups: (a) those that spread some light over the amazing complexity of the lattice, and (b) those that describe some well-behaved parts of the lattice. More precisely:

- By considering the connected rotation-annihilation method used to build involutive left-continuous continuous t-norm, we have proposed a possible way to decompose MTL-chains and we have studied some particular cases of this decomposition. This has resulted in an extension of the theory of perfect, local and bipartite algebras formerly used in varieties of MV and BL-algebras, to the variety of all MTL-algebras.
- Perfect IMTL-algebras have been proved to be exactly (modulo isomorphism) the disconnected rotations of prelinear semihoops (a particular case of the decomposition as connected rotation-annihilation).
- The lattice of varieties generated by perfect IMTL-algebras has been proved to be isomorphic to the lattice of varieties of prelinear semihoops.
- A decomposition theorem of every MTL-chain as an ordinal sum of indecomposable prelinear semihoops has been proved. Since all IMTL-chains are indecomposable and, as the previous item states, we have the complexity of all the lattice of varieties inside the involutive part, the description of all indecomposable prelinear semihoops seems to be a hopeless task.

- A particular class of indecomposable MTL-chains has been studied, namely weakly cancellative chains. We have studied the logics associated to these chains.
- We have studied the varieties of MTL-chains where a weak form of contraction, the so-called n -contraction law, holds. This condition yields a global form of Deduction Detachment Theorem and allows to prove several properties of their related logics.
- We have focused on a particular subvariety of 3-contractive MTL-algebras, namely Weak Nilpotent Minimum, obtaining a number of results on axiomatization of their subvarieties, local finiteness, generic chains and standard completeness.
- Finally, we have studied the expansions of t-norm based logics with truth-constants and their standard completeness properties.

During the investigation many interesting problems have arisen. Some of them are still open and will be the object of future research:

1. The characterization of chains which are indecomposable as ordinal sum seems to be a hopeless task, but maybe some advances can be done regarding indecomposable chains w.r.t. the connected rotation-annihilation construction.
2. The decidability of Π and Π MTL has been solved, but for the rest of weakly cancellative fuzzy logics remains open.
3. Is the canonical FSSC true for expansions of Łukasiewicz logic with irrational truth-constants?
4. In Chapter 10 we have proved that for almost all the t-norms in **CONT-fin** \cup **WNM-fin** (except for one case that remains open), the canonical SC turns out to be equivalent to the canonical SC restricted to evaluated formulae, and to the canonical FSSC restricted to positively evaluated formulae. Can this be extended to a bigger class of left-continuous t-norms?
5. Are all varieties of n -contractive MTL-algebras locally finite?
6. FEP, FMP and decidability for \mathbb{S}_n IMTL.
7. Standard completeness for $\Omega(C_n\text{WCMTL})$.
8. For all the axiomatic expansions of MTL considered so far there are two standard completeness properties that turn out to be equivalent: the SC and the FSSC. Is this true in general in the scope of algebraizable axiomatic expansions of MTL?
9. Another pair of properties has revealed to be equivalent for all the considered varieties: the FEP and the FMP. Is this true in general in the scope of algebraizable axiomatic expansions of MTL?

Index

- $Bp(x)$, 68
- $\Omega(\mathbb{L})$, 90
- $\bigoplus_{i \in I} \mathcal{A}_i$, 51
- \mathbb{BA} , 34
- \mathbb{BP}_0 , 68
- $\mathbb{BP}_0^{+\omega}$, 78
- \mathbb{BP}_0^{+n} , 78
- \mathbb{Fi} , 42
- \mathbb{IBP}_0 , 68
- \mathbb{IBP}_0^{+1} , 78
- \mathbb{L} , 34
- \mathbb{MTL} , 34
- \mathbb{RL} , 33
- $\mathcal{A} \odot \mathcal{B}$, 56
- \mathcal{A}^* , 54
- \mathcal{A}^\star , 55
- $L_*(\mathcal{C})$ -algebra, 142
- CONT**, **CONT-fin**, 144
- WNM**, **WNM-fin**, 119
- algebra, 11
 - congruent permutable, 16
 - finitely subdirectly irreducible, 15
 - of formulae, 12
 - partially embeddable, 18
 - product, 14
 - quotient, 13
 - reduced product, 14
 - semisimple, 15
 - simple, 13
 - subdirect product, 15
 - subdirectly irreducible, 15
 - ultraproduct, 15
- algebraic language, 11
- algebraic operators, 15
- algebraic semantics, 21
- Archimedean class, 46
- axiom, 20
 - n -contraction, 103
 - n -simplicity, 105
 - book-keeping, 142
 - cancellation, 30, 85
 - contraction, 30
 - divisibility, 30
 - involution, 30
 - prelinearity, 31
 - pseudocomplementation, 30
 - weak cancellation, 86
- Birkhoff Theorem, 16
- Bivalence Principle, 1
- canonical standard $L_*(\mathcal{C})$ -chain, 142
- comparative truth, 139
- congruence, 13
 - generated, 13
 - principal, 13
- congruence extension property (CEP), 18
- connectives
 - additive, 31
 - multiplicate, 31
- consequence relation, 19
- constant interval, 119
- Deduction-Detachment Theorem, 23
 - Local, 24, 42
- defining equations, 21
- drastic product, 76
- embedding, 13
- equation, 16

- equational consequence, 20
- equationally definable principal congruences property (EDPC), 17
- equivalence formulae, 21
- equivalent algebraic semantics, 21
- evaluated formulae, 164
- expansion, 21
 - axiomatic, 21
 - conservative, 21
- extension, 21
 - axiomatic, 21
- filter, 14
 - generated, 42
 - implicative, 41
 - of MTL-algebra, 41
 - principal, 42
 - proper, 14
- finite embeddability property (FEP), 18
- finite model property (FMP), 18
- fragment, 21
- fusion, 31
- fuzzy set, 2
- Gödel negation, 37
- Goguen implication, 37
- Hilbert-style calculus, 20
- homomorphic image, 13
- homomorphism, 13
- hoop, 45
 - basic, 45
 - cancellative, 45
 - Wajsberg, 45
- idempotent element, 42
- indecomposable totally ordered semihoop, 52
- inference rule, 20
- interval associated to an involutive element, 118
- involutive interval, 119
- isomorphism, 13
- Jónsson's Lemma, 17
- lattice, 12
 - bounded, 12
 - complete, 13
 - distributive, 12
 - order, 13
- lexicographic product, 24
- logic
 - n -contractive, 104
 - $L_*(C)$, 142
 - algebraizable, 21
 - BL, 29
 - decidable, 20
 - finitary, 20
 - finitely axiomatizable, 20
 - FL_{ew} , 31
 - G, 30
 - H_{BCK} , 31
 - many-valued, 32
 - ML, 30
 - MTL, 29
 - Product (II), 30
 - propositional, 19
 - sentential, 19
 - substructural, 31
 - t-norm based, 35
- membership function, 2
- MTL-algebra, 34
 - bipartite, 68
 - local, 71
 - locally finite, 72
 - order of an element, 67
 - peculiar, 72
 - perfect, 67
 - perfect plus n -points, 76
 - perfect plus fixpoint, 76
- MTL-chain
 - weakly cancellative, 86
- MTLH-algebra, 45
- MV-algebra, 34
- negative element, 43
- nilpotent element, 42
- ordinal sum
 - of t-norms, 50

- of totally ordered semihoops, 51
- partial embeddability property, 150
- Pavelka-style extension, 141
- positive element, 43
- positively evaluated formulae, 168
- propositional connective, 19
- propositional language, 19
- provability degree, 140
- quasiequation, 17
 - generalized, 17
- quasivariety, 17
 - equivalent quasivariety semantics, 21
- radical, 43
- representation
 - as subdirect product, 15
- residuated lattice, 32
- rotation
 - connected, 55
 - disconnected, 54
- rotation-annihilation
 - connected, 56, 75
 - disconnected, 55
- semihoop, 45
 - prelinear, 45
- Sorites Paradox, 1
- standard algebra of type F , 145
- standard completeness
 - canonical, 64
- standard completeness (SC), 59
 - finite strong (FSSC), 59
 - strong (SSC), 59
- strong finite model property (SFMP), 18
- strong negation function, 49
- subalgebra, 13
 - partial, 18
- substitution, 19
- subuniverse, 13
- theorem, 20
- totally decomposable MTL-chain, 53
- triangular t-norm (t-norm), 35
- truth-degree, 140
- ultrafilter, 14
- variety, 16
 - arithmetic, 16
 - congruent distributive, 16
 - congruent permutable, 16
 - locally finite (LF), 18
- Wajsberg algebra, 34
- weak negation function, 49
- WNM-chain
 - finite partition, 118
 - generic, 124
- zero divisor, 43

Bibliography

- [1] R. ADILLON AND V. VERDÚ. On a contraction-less intuitionistic propositional logic with conjunction and fusion, *Studia Logica* 65 (2000) 11–30.
- [2] P. AGLIANÓ, I. M. A. FERREIRIM AND F. MONTAGNA. Basic hoops: an algebraic study of continuous t-norms. To appear in *Studia Logica*.
- [3] P. AGLIANÓ AND F. MONTAGNA. Varieties of BL-algebras I: general properties, *Journal of Pure and Applied Algebra*, 181 (2003) 105–129.
- [4] C. ALSINA, E. TRILLAS AND L. VALVERDE. On some logical connectives for Fuzzy Set Theory, *J. Math. An. and Appl.* 93 (1983) 15–26.
- [5] R. AMBROSIO AND A. LETTIERI. A classification of bipartite MV-algebras, *Math. Japonica* 38, No. 1 (1993), 111–117.
- [6] ARISTOTLE. *The complete works of Aristotle. The revised Oxford translation*, Edited by Jonathan Barnes, Volume One, Bollingen Series LXXI, Princeton University Press, 1984.
- [7] M. BAAZ. Infinite-valued Gödel logics with 0 – 1 projections and relativizations, In *GÖDEL '96 - Logical foundations of mathematics, computer science and physics. Lecture Notes in Logic* 6 (1996), P. Hájek (Ed.), Springer Verlag, 23–33.
- [8] M. BAAZ, A. CIABATTONI AND F. MONTAGNA. Analytic Calculi for Monoidal T-norm Based Logic, *Fundamenta Informaticae* vol. 59, Number 4, pp. 315–332. 2004.
- [9] M. BAAZ, A. CIABATTONI AND C. FERMÜLLER. Hypersequent Calculi for Gödel Logics — a Survey, *Journal of Logic and Computation* vol. 13. pp. 835 - 861. 2003.
- [10] M. BAAZ, P. HÁJEK, D. ŠVEJDA AND J. KRAJÍČEK. Embedding logics into product logic, *Studia Logica* 61 (1998) 35–47.
- [11] P. BAHLS, J. COLE, N. GALATOS, P. JIPSEN AND C. TSINAKIS. Cancellative residuated lattices, *Algebra Universalis* 50 (2003) 83–106.

- [12] L. BĚHOUNEK AND P. CINTULA. Fuzzy Logics as the Logics of Chains, *Fuzzy Sets and Systems* 157 (2006) 604–610.
- [13] L.P. BELLUCE, A. DI NOLA AND A. LETTIERI. Local MV-algebras, *Rend. Circ. Mat. Palermo* (2) 42 (1993), 347–361.
- [14] G. BIRKHOFF. On the structure of abstract algebras, *Proc. Camb. Philos. Soc.* 31 (1935) 433–454.
- [15] G. BIRKHOFF. Subdirect unions in universal algebra, *Bulletin of the American Mathematical Society* 50 (1944) 764–768.
- [16] G. BIRKHOFF. *Lattice theory*, AMS Colloquim Publications, 1973.
- [17] W. J. BLOK AND I. M. A. FERREIRIM. On the structure of hoops, *Algebra Universalis* 43 (2000) 233–257.
- [18] W. J. BLOK AND D. PIGOZZI. On the structure of varieties with equationally definable principal congruences I, *Algebras Universalis* 15 (1982) 195–227.
- [19] W. J. BLOK AND D. PIGOZZI. Algebraizable logics, *Memoirs of the American Mathematical Society* 396, vol 77, 1989.
- [20] W. J. BLOK AND D. PIGOZZI. Local Deduction Theorems in Algebraic Logic, *Algebraic Logic* (Proc. Conf. Budapest 1988) (H. ANDREKA, J. D. MONK AND I. NEMETI, EDS) Colloq. Math. Soc. J. Bolyai 54 (1991) 75–105.
- [21] W. J. BLOK AND D. PIGOZZI. Abstract Algebraic Logic and the Deduction Theorem, To appear in *Bulletin of Symbolic Logic*.
- [22] W. J. BLOK AND C. J. VAN ALTEN. The finite embeddability property for residuated lattices, pocrimis and BCK-algebras, *Algebra Universalis* 48 (2002) 253–271.
- [23] D. BOIXADER, F. ESTEVA AND L. GODO. On the continuity of t-norms on bounded chains, *Proc. of IFSA 1999*, 476–479.
- [24] S. BURRIS AND H. P. SANKAPPANAVAR. *A course in Universal Algebra*, Springer Verlag, New York, 1981.
- [25] C. C. CHANG. Algebraic analysis of many valued logics, *Transactions of the American Mathematical Society* 88 (1958) 456–490.
- [26] C. C. CHANG. A new proof of the completeness of the Lukasiewicz axioms, *Transactions of the American Mathematical Society* 93 (1959) 74–80.
- [27] A. CIABATTONI, F. ESTEVA AND L. GODO. T-norm based logics with n -contraction, *Neural Network World* 12 (2002) 453–460.

- [28] A. CIABATTONI, G. METCALFE AND F. MONTAGNA. Adding modalities to fuzzy logics. To appear in the book *Fuzzy Logics and Related Structures* (S. Gottwald, P. Hájek, U. Höhle, E.P. Klement, eds.), Elsevier, 2007.
- [29] R. CIGNOLI, F. ESTEVA, L. GODO AND F. MONTAGNA. On a class of left-continuous t-norms, *Fuzzy Sets and Systems* 131 (2002) 283–296.
- [30] R. CIGNOLI, F. ESTEVA, L. GODO AND A. TORRENS. Basic Fuzzy Logic is the logic of continuous t-norms and their residua, *Soft Computing* 4 (2000) 106–112.
- [31] R. CIGNOLI AND D. MUNDICI. Partial isomorphisms on totally ordered abelian groups and Hájek’s completeness theorem for Basic Logic, *Multiple Valued Logic*, special issue dedicated to the memory of Grigore Moisil, vol. 6 (2001) 89–94.
- [32] R. CIGNOLI AND A. TORRENS. An algebraic analysis of product logic. *Mult. Val. Logic* 5 (2000) 45–65.
- [33] R. CIGNOLI AND A. TORRENS. Free algebras in varieties of BL-algebras with a Boolean retract, *Algebra Universalis* 48 (2002) 55–79.
- [34] R. CIGNOLI AND A. TORRENS. Glivenko like theorems in natural expansions of BCK-logic, *Mathematical Logic Quarterly* 50(2) (2004) 111–125.
- [35] P. CINTULA. About axiomatic systems of product fuzzy logic, *Soft Computing* 5 (2001) 243–244.
- [36] P. CINTULA. Short note on the redundancy of axiom (A3) in BL and MTL, *Soft Computing* 9 (2005) 942–942.
- [37] P. CINTULA. *From Fuzzy Logic to Fuzzy Mathematics*. Ph.D. dissertation, Czech Technical University, Prague (Czech Republic), 2005.
- [38] P. CINTULA. Weakly Implicative (Fuzzy) Logics (I): Basic Properties, *Archive for Mathematical Logic* 45 (2006) 673–704.
- [39] A. H. CLIFFORD. Naturally totally ordered commutative semigroups, *Amer. L. Math.* 76 (1954) 631–646.
- [40] A. C. CLIMESCU. Sur l’équation fonctionnelle de l’associativité *Bull. École Polytechn. Iassy* 1 (1946) 1–16.
- [41] J. CZELAKOWSKI. *Protoalgebraic logics*, vol. 10 of Trends in Logic, Kluwer, Dordrecht 2001.
- [42] J. CZELAKOWSKI AND W. DZIOBIAK. Congruence distributive quasivarieties whose finitely subdirectly irreducible members form a universal class, *Algebra Universalis* 27 (1990) 128–149.

- [43] R. P. DILWORTH AND M. WARD. Residuated Lattices, *Transactions of the American Mathematical Society* 45 (1939) 335–354.
- [44] A. DI NOLA, MV-Algebras in the Treatment of Uncertainty, In *Proceedings of the International IFSA Congress*, Bruxelles 1991 (P. Löwen and E. Roubens, eds), Kluwer, Dordrecht, 1993, 123–131.
- [45] A. DI NOLA, F. LIGUORI AND S. SESSA. Using maximal ideals in the classification of MV-algebras, *Portugaliae Mathematica* 50 (1993) 87–102.
- [46] A. DI NOLA, S. SESSA, F. ESTEVA, L. GODO AND P. GARCIA. The variety generated by perfect BL-algebras: an algebraic approach in a fuzzy logic setting, *Annals of Mathematics and Artificial Intelligence* 35 (2002) 197–214.
- [47] M. DUMMETT. A propositional calculus with denumerable matrix, *The Journal of Symbolic Logic* 24 (1959) 97–106.
- [48] F. ESTEVA AND X. DOMINGO. Sobre negaciones fuertes y débiles en $[0, 1]$, *Stochastica* 4 (1980) 141–166.
- [49] F. ESTEVA, J. GISPERT, L. GODO AND F. MONTAGNA. On the standard and Rational Completeness of some axiomatic extensions of Monoidal t-norm Based Logic, *Studia Logica* 71 (2002) 199–226.
- [50] F. ESTEVA, J. GISPERT, L. GODO AND C. NOGUERA. Adding truth-constants to logics of continuous t-norm: axiomatization and completeness results, *Fuzzy Sets and Systems*, 158 (2007) 597–618.
- [51] F. ESTEVA AND L. GODO. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms, *Fuzzy Sets and Systems* 124 (2001) 271–288.
- [52] F. ESTEVA, L. GODO, P. HÁJEK AND F. MONTAGNA. Hoops and Fuzzy Logic, *Journal of Logic and Computation*, Vol. 13 No. 4 (2003) 531–555.
- [53] F. ESTEVA, L. GODO AND F. MONTAGNA. Equational characterization of the subvarieties of BL generated by t-norm algebras, *Studia Logica* 76 (2004) 161–200.
- [54] F. ESTEVA, L. GODO, P. HÁJEK AND M. NAVARA. Residuated fuzzy logic with an involutive negation, *Archive for Mathematical Logic* 39 (2000) 103–124.
- [55] F. ESTEVA, L. GODO AND C. NOGUERA. On rational weak nilpotent minimum logics, *Journal of Multiple-Valued Logic and Soft Computing* 12 (2006) 9–32.
- [56] F. ESTEVA, L. GODO AND C. NOGUERA. Survey on expansions of t-norm based logics with truth-constants. To appear in the book *Fuzzy Logics and Related Structures* (S. Gottwald, P. Hájek, U. Höhle and E.P. Klement eds.), 2007.

- [57] K. EVANS, M. KONIKOFF, J. J. MADDEN, R. MATHIS AND G. WHIPPLE. *Totally Ordered Commutative Monoids*. Semigroup Forum 62 (2001) 249–278.
- [58] I. M. A. FERREIRIM. *On varieties and quasivarieties of hoops and their reducts*, Ph.D. dissertation, University of Illinois at Chicago, 1992
- [59] J. FODOR. Nilpotent minimum and related connectives for fuzzy logic, *Proc. of FUZZ-IEEE'95*, 1995, pp. 2077–2082.
- [60] J. M. FONT, A. J. GIL, A. TORRENS AND V. VERDÚ. An infinite-valued Lukasiewicz logic that preserves degrees of truth. To appear in *Archive for Mathematical Logic*.
- [61] J. M. FONT AND R. JANSANA, *A general algebraic semantics for sentential logics*, Springer, 1996.
- [62] J. M. FONT, R. JANSANA AND D. PIGOZZI, A Survey of Abstract Algebraic Logic, *Studia Logica* 74 (2003) 13–97.
- [63] J. M. FONT, A. J. RODRÍGUEZ AND A. TORRENS. Wajsberg Algebras, *Stochastica* Vol. VIII, 1 (1984) 5–31.
- [64] L. FUCHS. *Partially Ordered Algebraic Systems*, Pergamon Press, Oxford, London, New York, Paris, 1963.
- [65] H. GAITÁN. Subdirectly irreducible MV-algebras, *Czechoslovak Mathematical Journal*, 53 (2003) 631–639.
- [66] N. GALATOS, P. JIPSEN, T. KOWALSKI AND H. ONO, *Residuated Lattices: an algebraic glimpse at substructural logics*, Studies in Logic and the Foundations of Mathematics 151, Elsevier, 2007.
- [67] N. GALATOS AND H. ONO. Algebraization, parametrized local deduction theorem and interpolation for substructural logics over FL, *Studia Logica* 83 (2006).
- [68] N. GALATOS AND C. TSINAKIS. Generalized MV-algebras, *Journal of Algebra* 283 (2005) 254–291.
- [69] A. GARCÍA-CERDAÑA, C. NOGUERA AND F. ESTEVA. On the scope of some formulas defining additive connectives in fuzzy logics, *Fuzzy Sets and Systems* 154 (2005) 56–75.
- [70] G. GERLA. *Fuzzy Logic: Mathematical Tools for Approximate Reasoning*, vol. 11 in Trends in Logic, Kluwer, Dordrecht, 2001.
- [71] J. GISPERT. Axiomatic extensions of the nilpotent minimum logic, *Reports on Mathematical Logic* 37 (2003) 113–123.

- [72] J. GISPERT AND A. TORRENS. Axiomatic extensions of IMT3 logic, *Studia Logica* 81 (2005) 311–324.
- [73] A. M. W. GLASS. *Partially Ordered Groups*, Series in Algebra, Vol. 7, World Scientific, 1999.
- [74] K. GÖDEL. Zum intuitionistischen Aussagenkalkül, *Anzeiger Akademie der Wissenschaften Wien, Math. naturwiss. Klasse* 69 (1932), 65–66.
- [75] J. A. GOGUEN. The logic of inexact concepts, *Synthese* 19 (1969) 325–373.
- [76] S. GOTTWALD. *A treatise on many-valued logics*, Studies in Logic and Computation, 9 Baldock: Research Studies Press, 2000.
- [77] S. GOTTWALD, A. GARCÍA-CERDAÑA AND F. BOU. Axiomatizing Monoidal Logic—A Correction, *Journal of Multiple-Valued Logic and Soft Computing* 9 (2003) 427–433.
- [78] G. GRÄTZER. *Universal Algebra*, Springer-Verlag, 1979.
- [79] P. HÁJEK. *Metamathematics of fuzzy logic*, vol. 4 in Trends in Logic, Kluwer, Dordrecht, 1998.
- [80] P. HÁJEK. Basic fuzzy logic and BL-algebras, *Soft Computing* 2 (1998) 124–128.
- [81] P. HÁJEK. Observations on the monoidal t-norm logic, *Fuzzy Sets and Systems* 132 (2002) 107–112.
- [82] P. HÁJEK AND P. CINTULA. On Theories and Models in Fuzzy Predicate Logics, *The Journal of Symbolic Logic* 71 (2006) 863–880.
- [83] P. HÁJEK, L. GODO AND F. ESTEVA. A complete many-valued logic with product-conjunction, *Archive for Mathematical Logic* 35 (1996) 191–208.
- [84] P. HÁJEK, T. HAVRÁNEK AND R. JIROUŠEK. *Uncertainty Processing in Expert Systems*, RC Press, Inc., Boca Raton, 1992.
- [85] P. HÁJEK AND P. PUDLÁK. *Metamathematics of First-Order Arithmetic*, Perspectives in Mathematical Logic, Springer, 1993.
- [86] L. HAY. Axiomatization of the infinite-valued predicate calculus, *The Journal of Symbolic Logic* 28 (1963) 77–86.
- [87] U. HÖHLE. Commutative, residuated l-monoids. In Höhle, U. and Klement. E.P. eds., *Non-Classical Logics and Their Applications to Fuzzy Subsets*, Kluwer Acad. Publ., Dordrecht (1995) 55–106.
- [88] R. HORČÍK. Standard Completeness Theorem for IIMTL, *Archive for Mathematical Logic* 44 (2005) 413–424.

- [89] R. HORČÍK. *Algebraic properties of fuzzy logics*, Ph. D. thesis, Czech Technical University in Prague, 2005.
- [90] R. HORČÍK. Decidability of cancellative extension of monoidal t-norm logic. To appear in *Logic Journal of the IGPL*, 2006.
- [91] R. HORČÍK. Alternative proof of standard completeness theorem for MTL, *Soft Computing* 11 (2007) 123–129.
- [92] R. HORČÍK AND P. CINTULA. Product Łukasiewicz Logic, *Archive for Mathematical Logic* 43 (2004) 477–503.
- [93] P. M. IDZIAK. *Lattice operations in BCK-algebras*, Ph. D. thesis, Jagiellonian University, 1983.
- [94] P. M. IDZIAK. Lattice operations in BCK-algebras, *Mathematica Japonica* 29 vol. 6 (1984) 839–846.
- [95] K. ISEKI. BCK-algebras with condition (S), *Mathematica Japonica* 24 (1979) 107–119.
- [96] S. JENEI. New family of triangular norms via contrapositive symmetrization of residuated implications, *Fuzzy Sets and Systems* 110 (1999) 157–174.
- [97] S. JENEI. Structure of left-continuous triangular norms with strong induced negations, (I) Rotation construction, *J. Appl. Non-Classical Logics* 10 (2000) 83–92.
- [98] S. JENEI. Structure of Girard monoids on $[0, 1]$, in: S. E. Rodabaugh, E. P. Klement (Eds.), *Topological and Algebraic Structures in Fuzzy Sets. A Handbook of Recent Developments in the Mathematics of Fuzzy Sets*, Kluwer Academic Publishers, Dordrecht, 2003, 277–308.
- [99] S. JENEI. On the structure of rotation-invariant semigroups, *Archive for Mathematical Logic*, 42 (2003) 489–514.
- [100] S. JENEI AND F. MONTAGNA. A proof of standard completeness for Esteva and Godo’s logic MTL, *Studia Logica* 70 (2002) 183–192.
- [101] S. C. KLEENE. On a notation for ordinal numbers, *The Journal of Symbolic Logic* 3 (1938) 150–155.
- [102] E. P. KLEMENT, R. MESIAR AND E. PAP. *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.
- [103] Y. KOMORI. Super Łukasiewicz propositional logics, *Nagoya Mathematical Journal* 84 (1981) 119–133.
- [104] T. KOWALSKI AND H. ONO. Residuated lattices: An algebraic glimpse at logics without contraction (preliminary report), 2001.

- [105] T. KOWALSKI. Semisimplicity, EDPC and discriminator varieties of residuated lattices, *Studia Logica* 77 (2004) 255–265.
- [106] W. KRULL. Axiomatische Begründung der allgemeinen Idealtheorie, *Sitzungsberichte der physikalisch medizinischen Societät der Erlangen* 56 (1924) 47–63.
- [107] J. LAMBEK. *Lectures on Rings and Modules*, Blaisdel, Waltham, Massachusetts - Toronto - London, 1966.
- [108] C. M. LING. Representation of associative functions, *Publ. Math. Debrecen* 12 (1965) 189–212.
- [109] J. LOSÍ AND R. SUSZKO. Remarks on sentential logics, *Indag. Math.* 20 (1958) 177–183.
- [110] J. ŁUKASIEWICZ AND A. TARSKI. Untersuchungen ber den Aussagenkalkl. *Comptes Rendus de la Societät des Sciences et des Lettres de Varsovie*, cl. iii 23 (1930) 1–21.
- [111] A. I. MAL'CEV. Several remarks on quasivarieties of algebraic systems, *Algebra i Logika Sem.* 5 (1966) 3–9.
- [112] K. MENGER. Statistical metrics, *Proc. Nat. Acad. Sci. USA* 8 (1942) 535–537.
- [113] C. A. MEREDITH. The dependence of an axiom of Łukasiewicz, *Transactions of the American Mathematical Society* 87 (1958) 54.
- [114] A. MESIAROVÁ. A note on the structure of n -contractive t-norms, *Proceedings of IFSA 2003*, 69–72.
- [115] G. METCALFE, N. OLIVETTI AND D. GABBAY. Analytic Proof Calculi for Product Logics, *Archive for Mathematical Logic* 43(7) 859–889, 2004.
- [116] F. MONTAGNA. Generating the variety of BL-algebras, *Soft Computing* 9 (2005) 869–874.
- [117] F. MONTAGNA, C. NOGUERA AND R. HORČÍK. On weakly cancellative fuzzy logics, *Journal of Logic and Computation* 16 (2006) 423–450.
- [118] P. S. MOSTERT AND A. L. SHIELDS. On the structure of semigroups on a compact manifold with boundary, *Annals of Math.* 65 (1957) 117–143.
- [119] C. NOGUERA, F. ESTEVA AND J. GISPERT. Perfect and bipartite IMTL-algebras and disconnected rotations of prelinear semihoops, *Archive for Mathematical Logic* 44 (2005) 869–886.
- [120] C. NOGUERA, F. ESTEVA AND J. GISPERT. On some varieties of MTL-algebras, *Logic Journal of the IGPL* 13 (2005) 443–466.

- [121] C. NOGUERA, F. ESTEVA AND J. GISPERT. On varieties generated by Weak Nilpotent Minimum t-norms, *Proceedings of Fourth EUSFLAT*, Barcelona, 2005, pp. 866-871.
- [122] C. NOGUERA, F. ESTEVA AND J. GISPERT. On n -contractive fuzzy logics: first results, *Proc. IPMU 2006*, pp. 1612-1619.
- [123] V. NOVÁK. On the syntactico-semantical completeness of first-order fuzzy logic. Part I. Syntax and Semantics, *Kybernetika* 26 (1990), 47-66.
- [124] V. NOVÁK. On the syntactico-semantical completeness of first-order fuzzy logic. Part II. Main Results, *Kybernetika* 26 (1990), 134-154.
- [125] V. NOVÁK, I. PERFILIEVA AND J. MOČKOŘ. *Mathematical Principles of Fuzzy Logic*, Kluwer Academic Pub., 1999.
- [126] H. ONO. Proof-Theoretic Methods in Non-classical Logic: an Introduction. Theories of Types and Proofs, Takahashi et al eds. *MSJ Memoirs* 2, 1998, 207-254.
- [127] H. ONO AND Y. KOMORI. Logics without the contraction rule, *The Journal of Symbolic Logic* 50 (1985) 169-201.
- [128] J. PAVELKA. On Fuzzy Logic I, II, III, *Zeitschrift für Math. Logik und Grundlagen der Math.* 25 (1979) 45-52, 119-134, 447-464.
- [129] E. POST. Introduction to a general theory of elementary propositions, *American Journal of Mathematics* 43 (1921) 163-185.
- [130] A. J. RODRÍGUEZ, *Un estudio algebraico de los cálculos proposicionales de Łukasiewicz*, PhD dissertation, Universitat de Barcelona, 1980.
- [131] A. ROSE AND J. B. ROSSER. Fragments of many-valued statement calculi, *Transactions of the American Mathematical Society* 87 (1958) 1-53.
- [132] P. SAVICKÝ, R. CIGNOLI, F. ESTEVA, L. GODO AND C. NOGUERA. On product logic with truth-constants, *Journal of Logic and Computation* 16 (2006) 205-225.
- [133] B. SCHWEIZER AND A. SKLAR. Statistical metric spaces, *Pacific J. Math.* 10 (1960) 313-334.
- [134] B. SCHWEIZER AND A. SKLAR. Associative functions and abstract semi-groups, *Publ. Math. Debrecen* 10 (1963) 69-81.
- [135] S. SESSA AND E. TURUNEN. Local BL-algebras, *Mult. Valued Logic* 6 (2001) 229-249.
- [136] E. TRILLAS. Sobre funciones de negación en la teoría de conjuntos difusos, *Stochastica* 3 (1979) 47-60.

- [137] E. TRILLAS AND L. VALVERDE. On some functionally expressable implications for fuzzy set theory, *Proc. of the 3rd International Seminar on Fuzzy Set Theory 1981*, Linz, 173–190.
- [138] P. VOJTÁŠ. Fuzzy logic programming, *Fuzzy Sets and Systems* 124 (2001) 361–370.
- [139] T. WILLIAMSON. *Vagueness*, Routledge, 1994.
- [140] L. A. ZADEH. Fuzzy sets, *Information Control* 8 (1965) 338–353.
- [141] L. A. ZADEH. Preface. In *Fuzzy Logic Technology and Applications*, R. J. Marks-II, Ed., IEEE Technical Activities Board, 1994.

Monografies de l'Institut d'Investigació en Intel·ligència Artificial

- Num. 1 J. Puyol, *MILORD II: A Language for Knowledge-Based Systems*
- Num. 2 J. Levy, *The Calculus of Refinements, a Formal Specification Model Based on Inclusions*
- Num. 3 L. Vila, *On Temporal Representation and Reasoning in Knowledge-Based Systems*
- Num. 4 M. Domingo, *An Expert System Architecture for Identification in Biology*
- Num. 5 E. Armengol, *A Framework for Integrating Learning and Problem Solving*
- Num. 6 J. L. Arcos, *The Noos Representation Language*
- Num. 7 J. Larrosa, *Algorithms and Heuristics for Total and Partial Constraint Satisfaction*
- Num. 8 P. Noriega, *Agent Mediated Auctions: The Fishmarket Metaphor*
- Num. 9 F. Manyà, *Proof Procedures for Multiple-Valued Propositional Logics*
- Num. 10 W. M. Schorlemmer, *On Specifying and Reasoning with Special Relations*
- Num. 11 M. López-Sánchez, *Approaches to Map Generation by means of Collaborative Autonomous Robots*
- Num. 12 D. Robertson, *Pragmatics in the Synthesis of Logic Programs*
- Num. 13 P. Faratin, *Automated Service Negotiation between Autonomous Computational Agents*
- Num. 14 J. A. Rodríguez, *On the Design and Construction of Agent-mediated Electronic Institutions*
- Num. 15 T. Alsinet, *Logic Programming with Fuzzy Unification and Imprecise Constants: Possibilistic Semantics and Automated Deduction*
- Num. 16 A. Zapico, *On Axiomatic Foundations for Qualitative Decision Theory - A Possibilistic Approach*
- Num. 17 A. Valls, *ClusDM: A multiple criteria decision method for heterogeneous data sets*
- Num. 18 D. Busquets, *A Multiagent Approach to Qualitative Navigation in Robotics*
- Num. 19 M. Esteva, *Electronic Institutions: from specification to development*
- Num. 20 J. Sabater, *Trust and Reputation for Agent Societies*

- Num. 21 J. Cerquides, *Improving Algorithms for Learning Bayesian Network Classifiers*
- Num. 22 M. Villaret, *On Some Variants of Second-Order Unification*
- Num. 23 M. Gómez, *Open, Reusable and Configurable Multi-Agent Systems: A Knowledge Modelling Approach*
- Num. 24 S. Ramchurn, *Multi-Agent Negotiation Using Trust and Persuasion*
- Num. 25 S. Ontañón, *Ensemble Case-Based Learning for Multi-Agent Systems*
- Num. 26 M. Sánchez, *Contributions to Search and Inference Algorithms for CSP and Weighted CSP*
- Num. 27 C. Noguera, *Algebraic Study of Axiomatic Extensions of Triangular Norm Based Fuzzy Logics*
- Num. 28 E. Marchioni, *Functional Definability Issues in Logics Based on Triangular Norms*
- Num. 29 M. Grachten, *Expressivity-Aware Tempo Transformations of Music Performances Using Case Based Reasoning*
- Num. 30 I. Brito, *Distributed Constraint Satisfaction*
- Num. 31 E. Altamirano, *On Non-clausal Horn-like Satisfiability Problems*

