Introduction

The aim of this thesis is to provide an analysis of definability of some classes of functions in the framework of many-valued logics based on triangular norms.

Triangular norms (t-norms for short) are binary commutative associative and monotone operations defined over the real unit interval $[0,1]$ having 1 as a neutral element (see [97], and Chapter 1). Left-continuity (i.e. left-continuity as a function on $[0,1]$) guarantees for a t-norm the existence of a unique *residuum*, i.e. a binary operation $\Rightarrow_*$ such that for all $x, y, z \in [0,1]$, $x{*}y \leq z$ iff $x \leq y \Rightarrow_* z$. T-norms and their residua provide a natural semantic interpretation for many-valued conjunctions and implications.

Many-valued logics have been long studied without relying on the concept of triangular norm. Important proposals (among others[1]) were carried out by Jan Łukasiewicz and Kurt Gödel. Łukasiewicz was the first to introduce a three-valued system in [100, 101] and an infinite-valued system in [102] which was proved to be complete independently by Rose and Rosser in [128] and by Chang in [18, 19].

Kurt Gödel published in 1932 an extremely short paper [66] concerning intuitionistic logic. Gödel introduced an infinite hierarchy of finitely-valued systems: his aim was to show that there is no finitely-valued propositional calculus that is sound and complete for intuitionistic logic. The infinite-valued version of his systems is now known as Gödel logic. This logic was shown to be complete by Dummet in [45].

In [75], Hájek suggested a new approach to many-valued logics as logics associated to t-norms[2]. The idea behind this interpretation consists in the fact that given a left-continuous t-norm $*$ and a propositional language L with set of connectives $\{\&, \to, \vee, \wedge, \overline{0}, \overline{1}\}$, we can define a $*$-*evaluation* v as a homomorphism from the algebra of formulas of L into the algebra $[0,1]_* = \langle [0,1], *, \Rightarrow_*, \max, \min, 0, 1\rangle$. Each formula is assigned a value from the real unit interval $[0,1]$, & is interpreted as the left-continuous t-norm $*$, and $\to$ is interpreted as the residuum $\Rightarrow_*$. In this way we can associate to a left-continuous t-norm a set $\mathcal{L}_*$ of formulas, called *the logic of the t-norm* $*$, defined as the set of all formulas φ such that for every $*$-evaluation v, $v(\varphi) = 1$. Similarly, it is

[1] Here we just mention the works of Łukasiewicz and Gödel, being the most relevant to the content of this thesis. See Hájek's monograph [75] for an historical introduction to many-valued logics.

[2] See also Gottwald's monograph [71].

possible to associate a logic $\mathcal{L}_{\mathcal{K}}$ to a class $\mathcal{K}$ of left-continuous t-norms, defined as the intersection of all $\mathcal{L}_*$ with $* \in \mathcal{K}$.

This new interpretation allowed to look from a different perspective at logics like Łukasiewicz logic and Gödel logic. Indeed, those systems can be regarded as logics associated to two of the fundamental continuous t-norms, i.e.: the Łukasiewicz t-norm $x *_l y = \max(x + y - 1, 0)$, and the minimum t-norm (or Gödel t-norm) $x *_g y = \min(x, y)$, respectively.

Hájek introduced in [75] the Basic Logic BL in order to provide an axiomatization of the tautologies common to all continuous t-norms. BL was proven to be the logic of continuous t-norms and their residua by Hájek in [74] and by Esteva, Cignoli, Godo and Torrens in [24]. Łukasiewicz and Gödel logics were shown to be extensions of BL, along with the Product logic (first introduced by Hájek, Godo and Esteva in [78]) that is the logic of the Product t-norm $x *_\pi y = x \cdot y$.

As mentioned above, left-continuity is a sufficient (and necessary) condition for a t-norm to guarantee the existence of a residual implication. Based on this consideration, Esteva and Godo introduced in [50] the Monoidal T-norm based Logic MTL, that is more general and weaker than BL (see Chapter 2). MTL was conjectured to be the logic of left-continuous t-norms and their residua. This conjecture was proven to be true by Jenei and Montagna in [95].

Logics based on left-continuous t-norms have a real-valued semantics, where connectives are interpreted by real-valued functions. Then, given any formula φ in the language of a logic $\mathcal{L}$, we can ask which function or class of functions can be associated to φ by the evaluation v. It is theoretically interesting to investigate and try to characterize classes of functions definable in a certain logic. This is especially important from the point of view of possible applications since we might be interested in using formulas whose interpretation corresponds to certain functions[3]. The investigation of the definability of classes of functions in t-norm based logics will be the central topic of this work.

i. Expansions by Independent Involutive Negations.

The expressive power of t-norm based logics strongly depends on the functions definable from the given t-norm (class of t-norms). However, sometimes we might need to have functions which cannot be obtained by composition of the available operators. In that case, the most direct strategy is to introduce new functions which enhance the expressive power of the logic. The case of triangular conorms (t-conorms for short) is a relevant example. T-conorms are binary commutative associative and monotone operations defined over the real unit interval $[0, 1]$ having 0 as a neutral element (see [97], and Chapter 1). A typical example of a t-conorm is given by the maximum $\max(x, y)$. T-conorms do not play a special role in logics based on t-norms (with the exception of the maximum), since in general they are not definable from other operators. Still, it would be important to have a logic where we can have t-conorms since they are

[3]As an example, we may cite the representation of neural networks by means of Łukasiewicz formulas introduced by Amato, Di Nola and Gerla in [4].

especially important from the point of view of possible applications, and they are a many-valued generalization of the classical Boolean disjunction.

A way of obtaining t-conorms in a t-norm based logic consists in exploiting the presence of a strong negation (see Chapter 1). A strong negation (also called involutive) is a one-variable function defined over the real unit interval that is strictly decreasing, continuous, and symmetric w.r.t. to the diagonal of the unit square. A classical example is the standard negation $n_s(x) = 1 - x$. Given a strong negation n and a t-norm $*$, we can always define a t-conorm $\diamond$ as $x \diamond y = n(n(x) * n(y))$. In t-norm based logics, negations are defined from the residuum as $n(x) = x \Rightarrow 0$, and so they strongly depend on the t-norm or on the class of t-norms. In general, given a left-continuous t-norm, the negation obtained from the residuum is not a strong negation. For instance, neither Product nor Gödel logic has an involutive negation, while in Łukasiewicz logic we immediately have the standard negation n_s. Clearly, this implies that neither in BL nor in MTL the negation will generally behave as an involutive negation.

Hájek proposed in [75, p.274] the following problem:

> "Investigate the extension of Basic Logic with a new negation satisfying the double negation axiom [...], i.e. the logic of continuous t-norms and t-conorms."

A first attempt to solve this problem was given by Esteva, Hájek, Godo and Navara in [51], where the authors expanded logics based on continuous t-norms without zero-divisors (see Chapter 1) by means of an independent involutive negation. However, they did not introduce a general treatment, but they rather dealt with particular logics.

A first general completeness theorem was given by Haniková in [73], but, once again, that results concerned only logics based on continuous t-norms without zero divisors.

Here we will deal with left-continuous t-norms, and our aim will be to give a general method for introducing an involutive negation independent from the t-norm. This will allow to define in a given logic (a class of) t-conorms from the given (class of) t-norm(s).

ii. Definability in the Logic $\text{Ł}\Pi\frac{1}{2}$.

$\text{Ł}\Pi\frac{1}{2}$ certainly is the most powerful t-norm based logic since it combines the Łukasiewicz logic and the Product logic. The interesting feature of $\text{Ł}\Pi\frac{1}{2}$ is that the functions definable in it are strictly related to functions definable in the field of real numbers. Indeed, we will see that formulas of the universal theory of real closed fields can be faithfully translated into equations in $\text{Ł}\Pi\frac{1}{2}$. This means that the functions definable in real closed fields can be defined in $\text{Ł}\Pi\frac{1}{2}$.

Given this connection with real closed fields, it can be easily seen that functions definable in $\text{Ł}\Pi\frac{1}{2}$ are piecewise rational functions, i.e. supremum of fractions of polynomials with rational coefficients. Our aim will be to try to characterize classes of left-continuous t-norms definable by terms in $\text{Ł}\Pi\frac{1}{2}$-algebras and consequently by formulas in the logic $Ł\Pi\frac{1}{2}$. Clearly, only those left-continuous t-norms representable as piecewise rational functions will be definable.

Why will this analysis turn out to be fundamental? The reason is simple. $Ł\Pi\frac{1}{2}$ is well-known to be decidable and in PSPACE. If we take a logic of a certain (class of) t-norm(s) that is (are) definable in $Ł\Pi\frac{1}{2}$, then this logic can be directly interpreted in $Ł\Pi\frac{1}{2}$, and can inherit those properties (modulo a polynomial-time translation). In particular, a logic associated to a definable t-norm would immediately turn out to be decidable and in PSPACE, while a logic of a class of definable t-norms would be decidable. It would be then interesting to answer the following question: are the main t-norm based logics also logics of certain classes of definable t-norms? For instance, is BL the logic of definable continuous t-norms? Is MTL the logic of definable left-continuous t-norms? A positive answer to those questions would show that those logics do enjoy (some of) the above mentioned properties. $Ł\Pi\frac{1}{2}$ may therefore be viewed as a super-expressive t-norm based logic that covers many important t-norm based logics and consequently ensures their decidability and inclusion in PSPACE.

iii. Application to the Representation of Uncertainty.

Besides the theoretical interest from the logical, algebraic and functional point of view, one might ask if the investigation concerning functional definability might shed some light on other fields or have any application. In other words: how useful can be the results on definability? A first answer to that question will be given by the above mentioned transmission of decidability and computational properties from $Ł\Pi\frac{1}{2}$ to logics of definable t-norms.

Furthermore, we will also show an interesting application of the results on definability concerning the representation of uncertainty. Indeed, we will show how the expressive power given by the possibility of defining several real-valued functions in t-norm based logics will make them a general and powerful framework for representing measures of uncertainty.

Measures of uncertainty aim at formalizing the strength of our beliefs in the occurrence of some events by assigning to those events a degree of uncertainty. From the mathematical point of view a measure of uncertainty is a real-valued function that gives an event a value from the real unit interval $[0, 1]$. A well-known example is given by probability measures which try to capture our degree of confidence in the occurrence of events by real-valued assessments. Esteva, Hájek, and Godo proposed in [77, 67] a new interpretation of measures of uncertainty in the framework of t-norm based logics. Given a sentence as "The proposition φ is plausible (probable, believable)", its degrees of truth can be interpreted as the degree of uncertainty of the proposition φ. Indeed, the higher is our degree of confidence in φ, the higher the degree of truth of the above sentence will result. In some sense, the predicate "is plausible (believable, probable)" can be regarded as a modal operator over the proposition φ. Then, given a measure of uncertainty μ, we can define modal many-valued formulas $\kappa(\varphi)$, whose interpretation is given by a real number corresponding to the degree of uncertainty assigned to φ under μ. Furthermore, we can translate the peculiar axioms governing the behavior of an uncertainty measure into formulas of a certain t-norm based logic, depending on the operations we need to represent. An adequate analysis of functional definability will allow an adequate choice among several possible logics.

Previous particular results concerning the representation of measures of uncertainty were presented in several works. We can mention the treatment of probability measures, necessity measures and belief functions proposed by Esteva, Hájek, and Godo in [77, 75, 67], [77], and [68], respectively; the treatment of conditional probability proposed by the present author and Godo in [106, 69, 70]; the treatment of (generalized) conditional possibility and necessity given by the present author in [104, 105]; and finally the treatment of simple and conditional non-standard probability given by Flaminio and Montagna in [59].

Here, our aim will be to give a general and comprehensive treatment of the representation of measures of uncertainty. In particular, we will show how it is possible to represent classes of measures such as probabilities, lower and upper probabilities, possibilities and necessities. We will deal with both conditional and unconditional measures. Important properties of the functions of t-norm based logics will then be useful in order to prove relevant features of the classes of measures represented.

iv. Structure of the Work and Contributions.

This work is divided in three parts. Part I is devoted to providing background notions concerning t-norms and logics based on t-norms that will be needed in the following chapters. Part II focuses on functional definability and contains the main contributions of this thesis. Part III deals with applications and develops from a general point of view the representation of measures of uncertainty based on functional definability.

Chapter 1. We introduce the basic properties of triangular norms and negation functions. We also review the fundamentals of construction methods such as ordinal sum, rotation, annihilation and rotation-annihilation.

Chapter 2. In this chapter we focus on the fundamental algebraic and logical properties of the Monoidal T-norm based Logic MTL and its main schematic extensions. We review the basic results concerning several kinds of completeness, expansions by means of rational truth constants and the Delta connective, and combination of different t-norms. Finally we recall the essential properties of first-order expansions.

Chapter 3. We provide a general treatment concerning the expansion of members of the family of t-norm based logics by means of an independent involutive negation. We establish the basic requirements in order to obtain completeness w.r.t. classes of algebras of the related varieties, like linearly ordered algebras and algebras over the real unit interval, both for the propositional and the first-order case. This generalizes previous works by Esteva, Godo, Hájek and Navara, [51] and by Haniková [73] that dealt only with expansions for logics based on continuous t-norms without zero-divisors, and gives a (more general) solution to an open problem proposed by Hájek in [75].

Chapter 4. We show that, given an ordered field between the field of rational numbers and the field of real algebraic numbers, Boolean combinations of polynomial equations and inequalities with rational coefficients for such a field can be translated into equations over the related $\mathrm{Ł\Pi}\frac{1}{2}$-algebra. In particular, the universal theory of Real Closed Fields is shown to be interpretable in the equational theory of $\mathrm{Ł\Pi}\frac{1}{2}$-algebras.

An immediate generalization of a result proven by Montagna in [111] shows that the $\mathrm{Ł\Pi}\frac{1}{2}$-algebra associated to any real closed field generates the whole variety of $\mathrm{Ł\Pi}\frac{1}{2}$-algebras. In particular so does the $\mathrm{Ł\Pi}\frac{1}{2}$-algebra $\mathbb{A}\mathrm{Ł\Pi}\frac{1}{2}$ whose lattice reduct is the unit interval of the real algebraic numbers, and consequently the logic $\mathrm{Ł\Pi}\frac{1}{2}$ is finitely strongly standard complete w.r.t. evaluations over the real algebraic numbers. We show that $\mathbb{A}\mathrm{Ł\Pi}\frac{1}{2}$ is the smallest subalgebra of the $\mathrm{Ł\Pi}\frac{1}{2}$-algebra over the real unit interval generating the whole variety.

We answer an open problem raised by Montagna in [111], and show that the lattice of subvarieties of $\mathrm{Ł\Pi}\frac{1}{2}$-algebras has the cardinality of the continuum.

We characterize sets definable by $\mathrm{Ł\Pi}\frac{1}{2}$-terms, and show that they are sets defined by Boolean combinations of polynomial equations and inequalities with rational coefficients (called $\mathbb{Q}$-semialgebraic sets).

We show that there is a polynomial-time reduction of the theory of real closed fields to $\mathrm{Ł\Pi}\frac{1}{2}$. This means that the universal theory of real closed fields and the equational theory of $\mathrm{Ł\Pi}\frac{1}{2}$ are both in PSPACE and that they are strictly linked from the computational point of view.

Chapter 5. We deal with definability of (left-continuous) t-norms both in the first-order theory of real closed fields and in the equational logic of $\mathrm{Ł\Pi}\frac{1}{2}$-algebras. We begin by giving negative results concerning left-continuous t-norms. In particular, we show that left-continuous t-norms with a set of infinite isolated discontinuity points, and with a dense set of discontinuity points are not definable. We provide a complete characterization of definable continuous t-norms, proving that a continuous t-norm is definable iff it is representable as a finite ordinal sum. We also give a complete characterization of definability of weak nilpotent minimum t-norms, showing that they are definable iff the induced negation has a finite number of discontinuity points. Moreover, we also show that the class of definable left-continuous t-norms is closed under constructions like Annihilation, Rotation and Rotation-Annihilation (under certain conditions).

We show that the logics MTL, SMTL, IMTL, WNM, BL, and SBL are finitely strongly standard complete w.r.t. to the related classes of t-norms definable in $\mathrm{Ł\Pi}\frac{1}{2}$.

We show that every logic that is complete w.r.t. a left-continuous t-norm definable in $\mathrm{Ł\Pi}\frac{1}{2}$ is in PSPACE. Moreover, we prove that every finitely axiomatizable logic that is complete w.r.t. a class of definable left-continuous t-norms is decidable.

Chapter 6. We give a general treatment for the representation of uncertainty in the framework of t-norm based logics. We will establish general conditions for a logic of uncertainty to be complete, and provide a general discussion about such a representation. We will deal with both simple measures and conditional measures of uncertainty. Previous results concerning the representation of probability, possibility and necessity measures (see [77, 75]) will be a direct consequence of our approach. Moreover, we will also provide a logical treatment of both lower and upper conditional probabilities.

We show that for logics with rational truth constant it is possible to define suitable theories whose consistency is tantamount to the coherence of the related assessment of uncertainty. Given certain functional properties of the logics, such assessments can be showed to be compact.

Chapter 7. We lay out a list of open problems related to the content of this work which deserve further investigation.

Appendix A. We will review the basic properties of uninorms (i.e. a generalization of both t-norms and t-conorms) and of their logics. We will extend the previous results concerning definability to several classes of left-continuous conjunctive uninorms. Furthermore, we will prove completeness of the logics UML and BUL w.r.t. the related classes of definable left-continuous conjunctive uninorms.

Appendix B. In this appendix we briefly review some basic algebraic notions used in the text.

v. Publications.

Many of the results contained in this work appeared or will appear in the following international publications.

The general treatment of the addition of an independent involutive negation is basically contained in the following papers:

- Flaminio T. and Marchioni E.: T-norm based logics with an independent involutive negation. *Fuzzy Sets and Systems*, Vol. 157, Issue 24, 3125–3144, 2006.

- Flaminio T. and Marchioni E.: Extending the Monoidal T-norm based Logic with an independent involutive negation. In *Proceedings of the 4th EUSFLAT Conference*, Barcelona (Spain), 860–865, 2005.

All the results about $Ł\Pi\frac{1}{2}$ contained in Chapter 4 and Chapter 5 can be found in:

- Marchioni E. and Montagna F.: On triangular norms and uninorms definable in $Ł\Pi\frac{1}{2}$. *International Journal of Approximate Reasoning*, in print, 2007.

- Marchioni E.: Ordered fields and $Ł\Pi\frac{1}{2}$-algebras. Submitted.

- Marchioni E. and Montagna F.: Complexity and definability issues in $Ł\Pi\frac{1}{2}$. *Journal of Logic and Computation*, Vol. 17, Number 2, 311–331, 2007.

- Marchioni E. and Montagna F.: A note on definability in $Ł\Pi\frac{1}{2}$. In *Proceedings of the 11th IPMU International Conference*, Paris (France), 1588–1595, 2006.

The general treatment for the representation of uncertainty measures in the framework of t-norm based logics is contained in the following works:

- Godo L. and Marchioni E.: Theories of uncertainty as modal theories in t-norm based logics. In preparation.

- Marchioni E.: Uncertainty as a modality over t-norm based logics. In *New Dimensions in Fuzzy Logic and Related Technologies*, Proceedings of the 5th EUSFLAT Conference, Ostrava (Czech Republic), 169–176, 2007.

Specific results concerning conditional probability and possibility can be found in:

- Marchioni E.: Possibilistic conditioning framed in fuzzy logics. *International Journal of Approximate Reasoning*, Vol. 43, Issue 2, 133–165, 2006.

- Godo L. and Marchioni E.: Reasoning about coherent conditional probability in a fuzzy logic setting. *Logic Journal of the IGPL*, Vol. 14, Number 3, 457–481, 2006.

- Marchioni E.: A logical treatment of possibilistic conditioning. In *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, Lecture Notes in Artificial Intelligence 3571, 701–713, Springer-Verlag, Berlin-Heidelberg, 2005.

- Godo L. and Marchioni E.: Reasoning about coherent conditional probability in the fuzzy logic FCP(ŁΠ). In *Proceedings of the Workshop on Conditionals, Information and Inference*, Ulm (Germany), 1–16, 2004.

- Marchioni E. and Godo L.: A logic for reasoning about coherent conditional probability: A modal fuzzy logic approach. In *Logics in Artificial Intelligence*, Lecture Notes in Artificial Intelligence 3229, 213–225, Springer-Verlag, Berlin-Heidelberg, 2004.

