Introduction

The contents of this book are placed in the framework of *many-valued logics* from two different but convergent perspectives: that of *substructural logics* and that of *mathematical fuzzy logic*. The present work is a contribution to the study of both the fragments of substructural logics and the fragments of fuzzy logics based on triangular norms, pseudonorms, uninorms, and also weakly implicative fuzzy logics. Mainly, we study some fragments without implication of the basic intuitionistic substructural logics. Since the formal systems belonging to mathematical fuzzy logic are axiomatic extensions of a basic substructural logic, one of the motivations of our study is to provide a base for the analysis of the mentioned fragments for substructural and fuzzy logics.

A *substructural logic* is a logic admitting a presentation in terms of sequents obtained by either dropping or restricting some of the structural rules of either intuitionistic or classical logic. The development of this field of research is intimately related to the book *Substructural Logics* [DSH93], edited by Kosta Došen and Peter Schroeder-Heister, and published in 1993. This book was the first monograph devoted to this family of logics. In the preface, the authors point out that the name *Substructural Logics* was used for first time at the *Seminar für natürlich-sprachliche Systeme* of the Tübingen University held in October 1990. The scope of substructural logics includes, among other logics, the following ones:

- *Intuitionistic Logic* is a substructural logic since it can be obtained by restricting the *weakening* rule of Classical Logic to sequents with the empty sequence or a sequence with only one formula in the consequent.
- *Relevance Logic*, that rejects the rule of *weakening*.
- The so-called *logics without contraction*, as the system H_{BCK} defined by Ono and Komori [OK85], commonly known as *Monoidal Logic* inside the mathematical fuzzy logic community.
- *Lineal Logic* (Girard, [Gir87]), that rejects both, the *weakening* and the *contraction* rules.
- *Lambek Calculus* [Lam58] for the analysis of syntactic structures, that rejects the rules of *exchange*, *weakening* and *contraction*.

From the early 90's the substructural logics have been a field of growing activity. Among the monographs devoted to a systematic study of these logics, the more prominent ones are those by Greg Restall [Res00], Francesco Paoli [Pao02] and, specially, the recent book *Residuated Lattices: An Algebraic Glimpse at Substructural Logics* by Galatos et altri [GJKO07].

Concerning research on fuzzy logics, the book *Metamathematics of Fuzzy Logic* [Háj98], by Petr Hájek, published in 1998, is the first book devoted to the study of continuous t-norm based fuzzy logics. This book introduces a systematic analysis of the deductive systems and the algebraic structures of fuzzy logics seen as many-valued logics. The author's goal is to show that fuzzy logic, as the logic of the vague propositions, admits a formal foundation. Thus the so called *fuzzy inference* may be seen as logical deduction. A logic is *fuzzy* and *t-norm based* if it is sound and complete with respect to the algebras over [0,1] given by a family of left continuous t-norms. The t-norm based fuzzy logics are extensions of the *Monoidal* T*-norm based fuzzy Logic* (MTL, for short) introduced by Francesc Esteva and Lluís Godo in [EG01]. MTL is the weaker of this family of logics and it can be obtained as an axiomatic extension of the Monoidal Logic (Höhle [Höh95]), which is a logic without contraction in the sense of [OK85] and therefore a substructural logic because –as Esteva, Godo, and García-Cerdaña point out in [EGGC03]– it coincides with the logic associated to the *Full Lambek calculus with exchange and weakening* ($\mathbf{FL}_{ew}$*-logic*) of Hiroakira Ono [Ono90, Ono93]. In other words, the Monoidal Logic is equal to the logic associated, as external deductive system, with the sequent calculus $\mathbf{FL}_{ew}$, and it is substructural because $\mathbf{FL}_{ew}$ can be obtained by dropping the structural rule of contraction of a version in terms of sequents of the Intuitionistic Logic. The article [EGGC03] is the first work in which there are pointed out the connections among both hierarchies of logics, substructural and fuzzy, by adding and combining different axioms used in both traditions.

On the other hand, inside the general framework of fuzzy logics, there are also logics such as *pseudo t-norms* based logics (see [Háj03b, Háj03a, JM03]), *uninorm* based logics (see [GM07, MM07]), or *weakly implicative fuzzy logics* (see [Cin04, Cin06]), that are also extensions of more general substructural logics. In the present work, we analyze the fragments without implication in the general framework of **FL**, that is, the most general basic intuitionistic substructural logic (without *exchange*, *weakening* and *contraction*). The analysis of the fragments of a logic is important in order to characterize the contribution of each connective to the general properties of that logic. The study of the fragments *with* implication of the substructural logics has received attention in the literature (see for example [vAR04] and the references that are quoted there). Concerning the fragments *with* implication of the logics based on t-norms, an essential work is [CHH07]. Nevertheless, the study of the fragments *without* implication of both, substructural and fuzzy logics, has not received attention in the literature. One exception of this fact is the study of the fragments in the languages $\langle \vee \rangle$, $\langle \wedge \rangle$, $\langle \vee, \wedge \rangle$ and $\langle \vee, \wedge, \neg \rangle$ of the intuitionist logic (see [PW75, DP80, FGV91, FV91] for the fragments in the languages $\langle \vee \rangle$, $\langle \wedge \rangle$, $\langle \vee, \wedge \rangle$ and [RV93, RV94] for the $\langle \vee, \wedge, \neg \rangle$-fragment). A first step towards a general study of the fragments without implication of the substructural logics can be found in the papers [BGCV06, AGCV07]. In [BGCV06] the authors analyzed the fragments in the languages $\langle \vee, *, \neg, 0, 1\rangle$ and $\langle \vee, \wedge, *, \neg, 0, 1\rangle$ of the logic associated to the calculus $\mathbf{FL}_{ew}$ without contraction. In [AGCV07] the fragments corresponding to the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$ and $\langle \vee, \wedge, 0, 1\rangle$ are studied. The contents of the present monograph can be seen as a generalization of some the results of the papers mentioned above to a more general framework of substructural logics.

The present monograph corresponds to one of the parts of the doctoral dissertation of Àngel García-Cerdaña, *Lògiques basades en normes triangulars: una contribució a l'estudi dels seus aspectes subestructurals* [GC07], supervised by Francesc Esteva and Ventura Verdú, and defended on the 20th November of 2007 in the University of Barcelona.[1] The thesis can be divided in two research lines: one of them [GC07, Part II] analyzes the connections between *fuzzy logics based on t-norms* and the framework of *substructural logics.* The contributions of this part have been published in the papers [GGCB03, EGGC03, GCNE05]. The contributions of the second research line are in the field of basic intuitionistic substructural logics and they are the main contents of this monograph. In particular, we analyze algebraically some fragments without implications of that logics. Part of this research is already published by Bou, García-Cerdaña and Verdú in [BGCV06] and by Adillon, García-Cerdaña and Verdú in [AGCV07].

Outline of the monograph

The main contents of the present monograph is a general study of basic substructural systems (Part II), and an algebraic analysis of some free-implication fragments of basic intuitionistic substructural logics (Part III). The used tools stand within the framework of Algebraic Logic and, in some parts, in the area of Abstract Algebraic Logic and algebraizable Gentzen systems.

Part I consists of three chapters where concepts and basic results are introduced. Chapter 1 is devoted to the notions and preliminary results of Universal Algebra. In Chapters 2 and 3 we introduce the notions and basic results concerning deductive systems (i.e., sentential logics) and Gentzen systems, as well as the corresponding notions of Algebraic Logic.

Part II (Chapters 4 and 5) is a contribution to the systematization of concepts and results already known but disseminated in a disperse way in the literature of the field. In addition, this part also contains some contributions to the study of the basic substructural systems.

In Chapter 4 we recall the definitions of the basic substructural Gentzen systems $\mathcal{FL}_\sigma$ and its associated external deductive systems $\mathfrak{e}\mathcal{FL}_\sigma$. They are presented in a language with two implications and two negations. We define the notion of *mirror image* of a sequent and we prove the *mirror image principle* for the systems $\mathcal{FL}_\sigma[\Psi]$, where Ψ contains the two implications or the two negations. We characterize the sequential

[1] This thesis is available online at http://www.iiia.csic.es/~angel/PhDthesis-A-Garcia-C.pdf

Leibniz congruence of the theories of the systems $\mathcal{FL}[\Psi]$ and the Leibniz congruence of the theories of the external systems $\mathfrak{e}\mathcal{FL}[\Psi]$, where the language Ψ contains one of the implication connectives, and prove that these external systems are protoalgebraic. Finally we provide known Hilbert-style axiomatizations for the external deductive systems $\mathfrak{e}\mathcal{FL}_\sigma$.

In Chapter 5 we present and focus on the notions of *notational copy*, *definability of connectives*, *definitional expansion* and *definitional equivalence* in the context of Gentzen systems. We also obtain results related to these notions for certain classes of Gentzen systems including the systems $\mathcal{FL}_\sigma[\Psi]$. These results are used in order to formalize some notions and claims which receive an informal treatment in the substructural logics literature, such as *collapse* and *definability* of connectives in certain systems $\mathcal{FL}_\sigma[\Psi]$ or the comparison among the different versions of the same system.

Part III (Chapters 6 until 9) is dedicated to the study of some fragments *without* implication of the systems $\mathcal{FL}_\sigma$ and $\mathfrak{e}\mathcal{FL}_\sigma$.

In Chapter 6 we introduce the ordered, latticed and semilatticed algebraic structures that will constitute the semantic core of some of the fragments later studied in Chapter 9. After some basic notions and preliminary results, we introduce the notion of *pointed monoid*. A pointed monoid (ordered, semilatticed or latticed) is obtained by adding the constant symbol 0 to the type of similarity of a monoid (ordered, semilatticed or latticed): such symbol is interpreted as a fixed element but arbitrary of the universe of the structure. We define the varieties of algebras $\mathring{\mathbb{M}}^{s\ell}_\sigma$ and $\mathring{\mathbb{M}}^{\ell}_\sigma$, where subindex σ is a subsequence of the sequence ew_lw_rc and symbols e, w_l, w_r and c codify what we refer to as (algebraic) exchange, right-weakening, left-weakening and contraction properties, respectively. Such properties, which are expressed by quasi-inequations are equivalent, respectively, to the following properties: commutativity, integrality, 0-boundedness and increasing idempotency. In Chapter 9 we will state the connection among the varieties $\mathring{\mathbb{M}}^{s\ell}_\sigma$ and $\mathring{\mathbb{M}}^{\ell}_\sigma$ (subvarieties of $\mathring{\mathbb{M}}^{s\ell}$ and $\mathring{\mathbb{M}}^{\ell}$ defined by the equations codified by σ) and the fragments of the systems $\mathcal{FL}_\sigma$ and $\mathfrak{e}\mathcal{FL}_\sigma$ in the languages $\langle \vee, *, 0, 1\rangle$ and $\langle \vee, \wedge, *, 0, 1\rangle$. At the end of the chapter, we recall the notion of residuation and the definitions and properties of residuated lattices, $\mathbb{FL}$-algebras and $\mathbb{FL}_\sigma$-algebras.

In Chapter 7 we introduce the notion of *pseudocomplementation* in the framework of the *pointed po-monoids*. The notion of pseudocomplement with respect to the monoidal operation can be seen as a generalization of the same notion defined in the framework of the pseudocomplemented distributive lattices (see [BD74, Lak73]). We define the class $\mathbb{PM}^{\preccurlyeq}$ of the pseudocomplemented po-monoids, and the classes $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ of the semilatticed and latticed pseudocomplemented monoids. We show that the classes $\mathbb{PM}^{\preccurlyeq}$ can be defined by means a set of inequations and thus the classes $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ are varieties. We also analyze the case when the pseudocomplementation is with respect to the minimum element of the monoid. In the two last sections we study the class of weakly contractive pseudocomplemented monoids and the class of involutive peseudocomplemented monoids. The pseudocomplements constitute the algebraic counterpart of negations: in Chapter 9 we will state the connection between the varieties $\mathbb{PM}^{s\ell}_\sigma$ and

$\mathbb{PM}^{\ell}_{\sigma}$ (subvarieties of $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ defined by the equations codified by σ) with the fragments of the Gentzen system $\mathcal{FL}_{\sigma}$ and the associated external deductive system $\mathfrak{e}\mathcal{FL}_{\sigma}$ in the languages $\langle \vee, *, \backslash, ', 0, 1\rangle$ and $\langle \vee, \wedge, *, \backslash, ', 0, 1\rangle$.

In Chapter 8 two kind of constructions of a complete $\mathbb{FL}$-algebra from any $\mathbb{FL}$-algebra are considered: the *Dedekind-MacNeille completion* (DM-completion, in short) and the *ideal-completion* (see [Ono93, Ono03a]). Both constructions allow to build a complete $\mathbb{FL}$-algebra from the monoidal reduct of a $\mathbb{FL}$-algebra in such a way that this algebra is embeddable into its completion. We show that the method of the ideal-completion also works if we start from an algebra in $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$ or $\mathbb{PM}^{\ell}_{\sigma}$ and we obtain that every algebra of these classes is embeddable into a complete $\mathbb{FL}_{\sigma}$-algebra. These embeddings have as a consequence that the classes $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$ and $\mathbb{PM}^{\ell}_{\sigma}$ are the classes of all the subreducts of the algebras in the class $\mathbb{FL}_{\sigma}$. However, we prove that the DM-completion, which works for $\mathbb{FL}_{\sigma}$-algebras, does not work when starting from the monoidal reduct of an algebra of the classes $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$ or $\mathbb{PM}^{\ell}_{\sigma}$ because in these cases the construction of Dedekind-MacNeille does in general not produce a $\mathbb{FL}_{\sigma}$-algebra. The reason is that to carry out this construction the monoidal operation must be residuated and this is not always the case.

In Chapter 9 we study the fragments in the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$ and $\langle \vee, *, \backslash, ', 0, 1\rangle$ of the systems $\mathcal{FL}_{\sigma}$ and their associated external deductive systems $\mathfrak{e}\mathcal{FL}_{\sigma}$. We prove that the subsystems $\mathcal{FL}_{\sigma}[\vee, *, 0, 1]$, $\mathcal{FL}_{\sigma}[\vee, \wedge, *, 0, 1]$, $\mathcal{FL}_{\sigma}[\vee, *, \backslash, ', 0, 1]$ and $\mathcal{FL}_{\sigma}[\vee, \wedge, *, \backslash, ', 0, 1]$ are algebraizables, having as respective equivalent algebraic semantics the varieties $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$, $\mathbb{PM}^{\ell}_{\sigma}$. We also prove that the system $\mathcal{FL}_{\sigma}$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{FL}_{\sigma}$. Using these results and the ones concerning subreducts obtained in the previous chapter we obtain that the mentioned subsystems are fragments of $\mathcal{FL}_{\sigma}$, and that the corresponding external deductive systems are fragments of $\mathfrak{e}\mathcal{FL}_{\sigma}$. We also show that each system $\mathcal{FL}_{\sigma}$ is equivalent to its associated external deductive system. However, it is shown that the considered fragments are not equivalent to any deductive system. Moreover we show that $\mathfrak{e}\mathcal{FL}_{\sigma}[\vee, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_{\sigma}[\vee, \wedge, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_{\sigma}[\vee, *, \backslash, ', 0, 1]$ and $\mathfrak{e}\mathcal{FL}_{\sigma}[\vee, \wedge, *, \backslash, ', 0, 1]$ are not protoalgebraic but they have, respectively, the varieties $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$, $\mathbb{PM}^{\ell}_{\sigma}$ as algebraic semantics with defining equation $1 \vee p \approx p$. We also give decidability results for some of the fragments considered. In the last section we define the basic substructural systems with weak contraction $\mathcal{FL}_{\sigma\hat{c}}$ and characterize the fragments in the languages $\langle \vee, *, \backslash, ', 0, 1\rangle$ and $\langle \vee, \wedge, *, \backslash, ', 0, 1\rangle$ of these systems and their associated external deductive systems.

In Chapter 10 we analyze the fragments in the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$, and $\langle \vee, \wedge, 0, 1\rangle$ in the context of $\mathcal{FL}_{ew}$. We obtain as main result that each one of these fragments coincides with the fragment in the same language of classical logic. Since MTL, the most general t-norm based fuzzy logic, is an axiomatic extension of $\mathfrak{e}\mathcal{FL}_{ew}$, we have as a corollary that the fragments in these three languages of all the t-norm based fuzzy logics are equal to the corresponding fragments of classical logic.

Chapter 11 is devoted to conclusions and future work.